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# Born-Lertes Effect on the Dipolar Rotation of Liquids.

E. GROSSETTI

*Istituto di Fisica Sperimentale dell'Università - Napoli*

(ricevuto il 30 Maggio 1958)

**Summary.** — We have determined the rotation moments of some polar liquids in a rotating electrical field for frequency values which are comprised between 0.325 MHz and 8.5 MHz. The noticeable differences we have found between the experimental values and the theoretic ones are been imputed to a change of conductivity and dielectric constant owing to the frequency change.

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M. BORN (1) proved that in a spheric volume of liquid, having a radius equal to $a$, with $\eta$ as viscosity coefficient and with molecules, having $M$ for electrical moment, submitted to a rotating electrical field, having $E$ for intensity and $\omega$ for pulsation, a mechanic moment $L$ is generated the value of which is determined by the following expression:

$$L = \frac{4}{3}\pi a^3 \eta \omega \left[\frac{ME}{kT}\right]^2 .$$

In view of the fact that the above expression presents some interesting aspects, particularly when the values of $\omega$ are not too high, some measurements were made similar to those that had already been taken by LERTES (2). Along the same lines as what had been done by the mentioned author and in order

(1) M. BORN: *Zeits. f. Phys.*, **1**, 221 (1920).
(2) P. LERTES: *Phys. Zeits.*, **3-4**, 315 (1921).

to obtain the rotating electrical field for frequencies between 0.325 MHz and 3.0 MHz a bridge system was used of the type illustrated in Fig. 1.

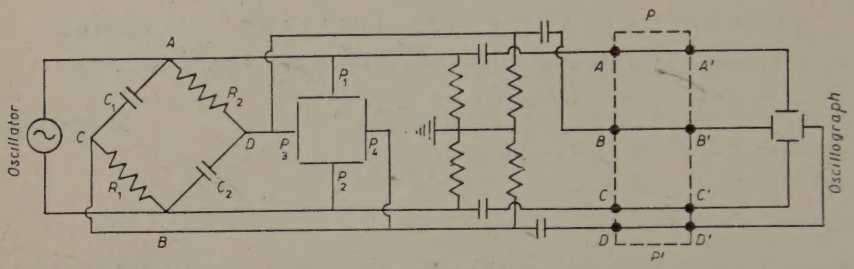


Fig. 1.

The radio frequency alternated voltage produced by the oscillator was conveyed to the two vertices of a bridge formed by resistances and capacities. If for $R$ and for $C$ values are selected that are capable of satisfying the following condition:

$$R_1 = R_2 = \frac{1}{C_1\omega} = \frac{1}{C_2\omega},$$

in which $\omega$ stands for the pulsation of the electric rotating field, at the four points $P_1$, $P_2$, $P_3$, $P_4$ alternated voltages are obtained that are by 90° out of phase with each other.

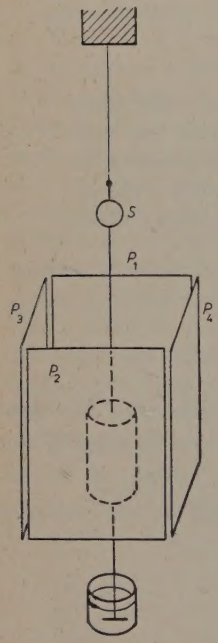


Fig. 2.

The points $A$, $B$ and $C$, $D$ were respectively linked to the four metal plates $P_1$, $P_2$, $P_3$, $P_4$, between which the rotating electric field was obtained. The four plates set according to Fig. 2 were brass plates having a thickness of 0.2 cm, a surface of $(12 \times 7)$ cm² at a distance of 9.0 cm. At the center of this system of plates a cylinder-shaped container $C$, empty, made of « teflon » (an apolar substance) contained the liquid to be examined.

The field [3] in a point $P$ within the four plates (Fig. 3), supposing the interval between the electrodes at the vertices to be null, is furnished by the following expressions that remain valid till $\varrho^4$ is negligible in respect of 1.

$$\text{Field in } P \begin{cases} E_x = V_0[0.834 \cos \Omega t + 0.715\,\varrho^2 \cos(\Omega t + 2\varphi)], \\ E_y = V_0[0.834 \sin \Omega t - 0.715\,\varrho^2 \sin(\Omega t + 2\varphi)], \\ E = 0.834 V_0[1 + 0.857\,\varrho^2 \cos 2(\Omega t + \varphi)]. \end{cases}$$

[3] The calculation of the field in the mentioned conditions was carried out by Prof. E. Persico. We wish hereby to thank him most sincerely.

The radius of the cylindrical vessel containing the liquid is 1.5 cm and the distance between the electrodes is 9.0 cm. Therefore referring these values to unit; that is to say assuming $4.5 = 1$, a point at a distance 1.5 cm from the center of the four plates is equivalent to a point placed at a distance 0.33 cm, for which $\varrho^4$ is negligible in respect of 1, so that we may assume that the vessel containing the liquid is totally immersed in a homogeneous rotating field.

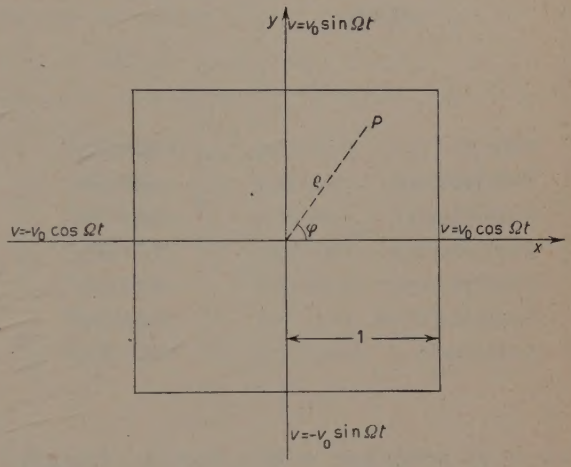


Fig. 3.

The teflon vessel was maintained suspended within the four plates and at their center by means of a wire (tungsten) having a diameter of $4.0 \cdot 10^{-3}$ cm and approximately 105 cm long. The unitarian torque moment resulted equal to 0.32 dyne × cm and was determined by means of the torque oscillations method. The readings of the rotation angles were carried out by the method of the scale and mirror, and at a distance of approximately 260 cm moments of $5.70 \cdot 10^{-5}$ dyne × cm could be observed. The teflon vessel did not present any rotation when immersed in the rotating electric field.

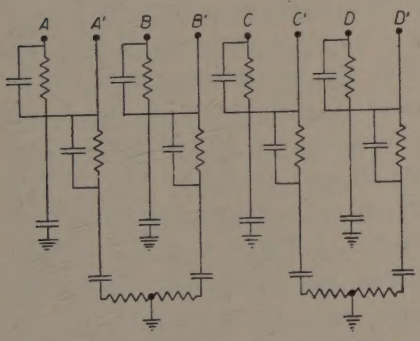


Fig. 4.

The measurement of the alternating fall in potential applied to the four plates of the rotating electric field was obtained connecting (Fig. 1) points $P_1$, $P_2$, $P_3$, $P_4$ to the opposed plates of a cathode ray oscillograph by means of the voltage separator $PP'$ having the ratio 1/10 (Fig. 4), so that it was possible to determine directly on the oscillograph the two differences of voltage that are out of phase with each other by 90°.

The verification of the 90° out of phase position of the two alternated voltages was obtained by controlling on the oscillograph the pattern traced by the cathodic spot, in consideration of the fact that by applying the two voltages separately it could be established whether they were equal, and having

TABLE I. – *Moment*

| Liquids | 0.325 MHz | | 0.650 MHz | |
|---|---|---|---|---|
| | $L_s$ | $L_t$ | $L_s$ | $L_t$ |
| Toluene . . . . . | $3.22 \cdot 10^{-4}$ | $1.83 \cdot 10^{-4}$ | $2.88 \cdot 10^{-4}$ | $9.10 \cdot 10^{-5}$ |
| Nitrobenzene . . . . . | $4.52 \cdot 10^{-4}$ | $5.56 \cdot 10^{-4}$ | $1.96 \cdot 10^{-4}$ | $4.27 \cdot 10^{-4}$ |
| Ethil alcohol . . . . . | $3.83 \cdot 10^{-3}$ | $6.2 \cdot 10^{-4}$ | $2.09 \cdot 10^{-3}$ | $3.39 \cdot 10^{-4}$ |
| Amyl alcohol . . . . . | $1.74 \cdot 10^{-3}$ | $6.43 \cdot 10^{-3}$ | $1.17 \cdot 10^{-3}$ | $3.47 \cdot 10^{-3}$ |
| Distilled water . . . . . | $2.95 \cdot 10^{-3}$ | $2.06 \cdot 10^{-3}$ | $1.57 \cdot 10^{-3}$ | $2.95 \cdot 10^{-3}$ |
| Methyl alcohol . . . . . | $8.70 \cdot 10^{-3}$ | $7.07 \cdot 10^{-3}$ | $5.54 \cdot 10^{-3}$ | $5.13 \cdot 10^{-3}$ |
| Acetic acid . . . . . | $2.26 \cdot 10^{-2}$ | $2.14 \cdot 10^{-2}$ | $1.04 \cdot 10^{-2}$ | $2.82 \cdot 10^{-2}$ |

set these conditions it was possible, from the form of the pattern, to make sure that the out of phase position was the required one.

For frequencies higher than 3.0 MHz the bridge method was abandoned because the capacity of the whole system of the four plates and all the connecting wires was no longer negligible and the 90° out of phase condition was obtained instead with the inductive system.

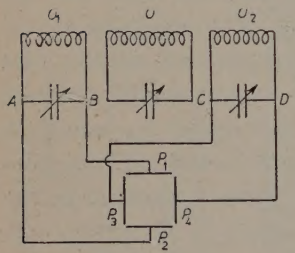


Fig. 5.

In this intent the circuit described in Fig. 5 was planned. At a suitable distance of the two circuits $O_1$, $O_2$ from the oscillating circuit $O$, in conditions of resonance, voltages in $A$ and $B$ are found to be out of phase by 90° in comparison to those in $C$ and $D$ and therefore the electric field between $P_1$, $P_2$, $P_3$, $P_4$ becomes a rotating field.

The frequencies used in this work were 0.325, 0.650, 1.4, 3.0, 8.5 MHz. For these experiments a cylindrical teflon vessel was employed. Therefore, in order the compare the experimental moments with the theoretical ones we can deduce from the Lampa's formula (see the following page) it is necessary to multiply the experimental values by a correcting coefficient so that we may reduce them to the ones referring to a spherical vessel having the same volume. The liquids examined were: ethyl alcohol, amyl alcohol, methyl alcohol, distilled water, toluene, nitro-benzene, acetic acid. Table I gives the values for the rotation moments at the various frequencies. The results are reproduced in Fig. 6. In this frequency spacing, however, it is not yet possible to use the Born-Leter's formula because there is another effect which dominates and mainly depends on the conductivity of the liquid. In fact,

dyne·cm.

| 1.4 MHz | | 3.0 MHz | | 8.5 MHz | |
|---|---|---|---|---|---|
| $L_s$ | $L_t$ | $L_s$ | $L_t$ | $L_s$ | $L_t$ |
| $1.66 \cdot 10^{-4}$ | $5.02 \cdot 10^{-5}$ | $2.95 \cdot 10^{-4}$ | $1.97 \cdot 10^{-5}$ | $6.0 \cdot 10^{-4}$ | $7.0 \cdot 10^{-6}$ |
| $1.39 \cdot 10^{-4}$ | $1.95 \cdot 10^{-4}$ | $1.04 \cdot 10^{-4}$ | $5.86 \cdot 10^{-5}$ | $1.57 \cdot 10^{-4}$ | $2.11 \cdot 10^{-5}$ |
| $9.60 \cdot 10^{-4}$ | $1.66 \cdot 10^{-4}$ | $8.20 \cdot 10^{-4}$ | $6.37 \cdot 10^{-5}$ | $8.70 \cdot 10^{-4}$ | $2.35 \cdot 10^{-5}$ |
| $5.65 \cdot 10^{-4}$ | $1.59 \cdot 10^{-3}$ | $4.55 \cdot 10^{-4}$ | $6.88 \cdot 10^{-4}$ | $7.57 \cdot 10^{-4}$ | $2.62 \cdot 10^{-4}$ |
| $7.12 \cdot 10^{-4}$ | $2.76 \cdot 10^{-3}$ | $3.92 \cdot 10^{-4}$ | $1.53 \cdot 10^{-3}$ | $8.05 \cdot 10^{-4}$ | $4.92 \cdot 10^{-4}$ |
| $2.75 \cdot 10^{-3}$ | $2.95 \cdot 10^{-3}$ | $1.18 \cdot 10^{-3}$ | $1.29 \cdot 10^{-3}$ | $1.61 \cdot 10^{-5}$ | $4.82 \cdot 10^{-3}$ |
| $3.30 \cdot 10^{-3}$ | $2.63 \cdot 10^{-2}$ | $3.78 \cdot 10^{-3}$ | $1.47 \cdot 10^{-2}$ | $1.57 \cdot 10^{-3}$ | $6.25 \cdot 10^{-3}$ |

at the lowest frequencies, an electric rotating field causes, the liquid being conductive, a charge displacement besides polarization. Indeed, LAMPA (4) proved that there is, besides the mechanical moment foreseen by BORN and caused by dipole orientation, a mechanical moment which depends just on the conductivity of the liquid. Under our conditions that is to say with liquids having those conductibility values and those pulsation values, it is not yet possible to clearly see the Born effect. But on the contrary, when a liquid has a conductibility $\lambda_i$ and a dielectric constant $\varepsilon_i$ there is the moment having the following expression (Lampa's formula) (referring to a spherical vessel):

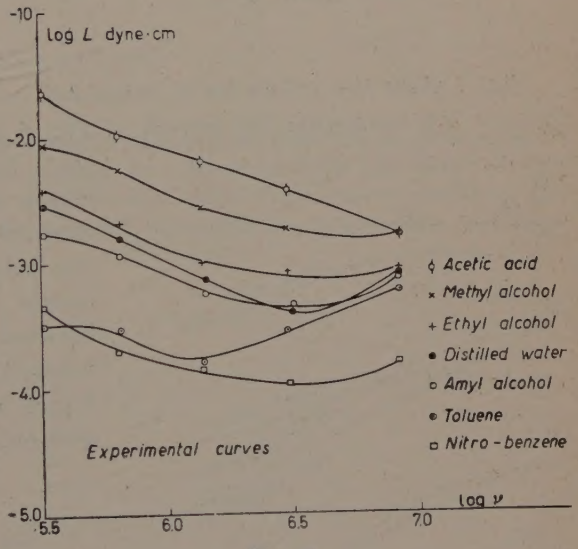


Fig. 6.

$$L = 36\pi E^2 R^3 \frac{\lambda_i \omega}{16\pi^2 \lambda_i^2 + \omega^2 (2 + \varepsilon_i)^2}.$$

(4) A. LAMPA: *Wien. Ber.*, **115** (2a), 1659 (1906).

Obviously, it is necessary to know the values of $\varepsilon_i$ and $\lambda_i$ in order to use this formula. We used as values of $\varepsilon_i$ the ones given by the tables of Landolt-Börnstein; the values of the conductivity, which more easily feel the effect of the origin of the liquid, have been measured with the classic method of the electrometer and the capacity. The Table II gives the values for the conductivity.

TABLE II.

| | $\lambda_i$ sper. in c·g s. |
|---|---|
| Toluene . . . . . | $1.01 \cdot 10^3$ |
| Nitro-benzene . . . . . | $2.59 \cdot 10^5$ |
| Ethyl alcohol . . . . . | $1.15 \cdot 10^5$ |
| Amyl alcohol . . . . . | $7.0 \cdot 10^5$ |
| Distilled water . . . . . | $3.31 \cdot 10^7$ |
| Methyl alcohol . . . . . | $4.97 \cdot 10^6$ |
| Acetic acid . . . . . | $4.60 \cdot 10^7$ |

Fig. 7 gives the values for $L$ calculated on the basis of these values of $\varepsilon_i$ and $\lambda_i$. By comparing the curves of Fig. 6 and Fig. 7 it is possible to see that the path of the experimental curves up to the frequency of 8.5 MHz does not differ to any great extent from the theoretic path. Fig. 8 shows the curves of the logarithms of the ratios between the moments measured experimentally and the theoretical ones for liquids studied on the basis of the logarithms of the frequencies. From these

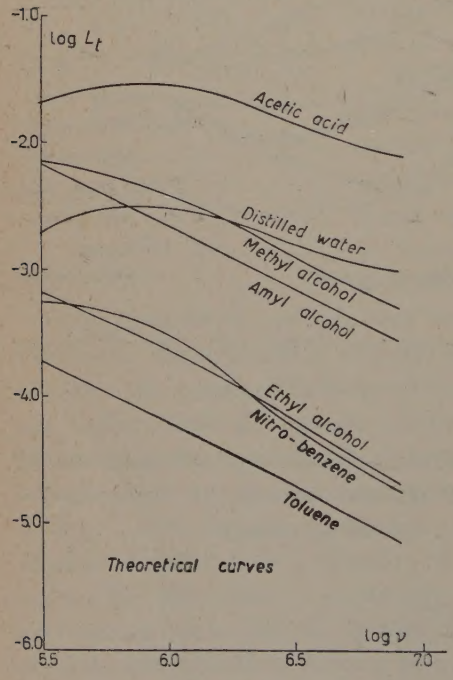


Fig. 7.

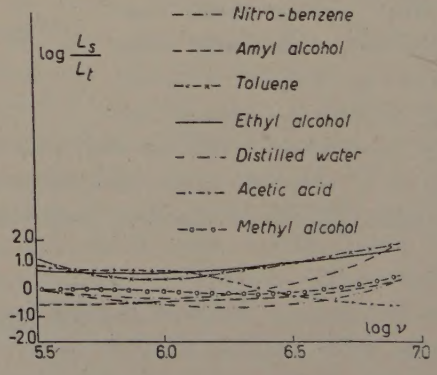


Fig. 8.

results we may observe that, generally speaking, Lampa's formula explains the results obtained. However the values for certain liquids present noticeable differences that are not absolutely to be imputed to errors in the measurings. They stand for a variation of $\lambda_i$ and of $\varepsilon_i$ from the values for frequencies null at the actual values.

In further papers the results related to higher frequencies, that is frequencies above 8.5 MHz, shall be discussed.

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I thank here very deeply Prof. A. CARRELLI who kindly gave me advices and suggestions and put the needful means at my disposal.

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## RIASSUNTO

In questo lavoro sono stati determinati i momenti di rotazione di diversi liquidi polari in campi elettrici rotanti per frequenze da 0.325 a 8.5 MHz. Per alcuni di essi la diversità riscontrata tra i valori sperimentali e quelli teorici sta a significare una variazione della conducibilità e della costante dielettrica al variare della frequenza.

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# High-Energy Electron Scattering from Nuclei with Finite Charge Distributions (*).

H. S. Valk

*Department of Physics, University of Oregon - Eugene, Oreg.*

(ricevuto il 3 Giugno 1958)

**Summary.** — By applying a relativistic extension of Molière's formula for the electron scattered amplitude to the particular case of potentials generated by radial charge distributions of finite extent, it is shown that the amplitude can be resolved into three distinct contributions: 1) the Mott point-charge scattered amplitude; 2) a correction to this arising from the finite radial cut-off and 3) a term characteristic of the charge distribution. The resulting expression is applied to the case of a uniform charge distribution for which an optical analogy is indicated.

## 1. – Derivation of the scattered amplitude.

It is the purpose of this note to examine the small-angle approximation of Schiff [1] and Malenka [2] for the special case of a potential arising from a finite and spherically-symmetric charge distribution. Under the conditions of validity of this approximation, it is shown that the scattered amplitude can be expressed in a form such that the part dependent on the nature of the charge distribution is isolated from those contributions describing the point-charge amplitude and its correction because of the finite radial cut-off. As a simple application of the resulting expression, we discuss the example of a uniform charge distribution. For this model, the term characteristic of the

(*) A preliminary report of this work was given at the 76th Meeting of the Oregon Section of the A.A.P.T.

(1) L. I. Schiff: *Phys. Rev.*, **103**, 443 (1956).

(2) B. J. Malenka: private communication.

distribution is shown to have a simple optical analogy related to the diffraction of spherical waves by a circular aperture.

It has been shown ([1-3]) that in the extreme relativistic case, the differential cross-section $(\mathrm{d}\sigma/\mathrm{d}\Omega)_r$ can be expressed in terms of its non-relativistic counterpart $(\mathrm{d}\sigma/\mathrm{d}\Omega)_{nr}$ by the relation

$$(\mathrm{d}\sigma/\mathrm{d}\Omega)_r = \cos^2(\theta/2)(E/mc^2)^2(\mathrm{d}\sigma/\mathrm{d}\Omega)_{nr}\,, \tag{1}$$

where $\theta$ is the scattering angle, $E$ is the total energy, and $mc^2$ is the rest energy of the electron. If we now restrict ourselves to small angles of scattering the non-relativistic cross-section can be approximated by

$$(\mathrm{d}\sigma/\mathrm{d}\Omega)_{nr} = |f(\theta)|^2\,, \tag{1a}$$

where

$$f(\theta) = ik\int_0^\infty p\,\mathrm{d}p J_0(qp)\{1-\exp[2i\delta(p)]\}, \tag{2}$$

and

$$\delta(p) = -(\hbar c)^{-1}\int_0^\infty V(p^2+z^2)\,\mathrm{d}z\,. \tag{3}$$

Here $k$ is the non-relativistic reduced wave number of the electron, $q\hbar$ is the momentum transfer, $V$ is the potential energy expressed in cylindrical coordinates, and the other symbols have their usual meanings. The scattered amplitude, Eq. (2), can be further reduced by inserting a convergence factor into the integrand and taking the limit as the factor goes to unity. In this way we obtain Molière's formula ([4]):

$$f(\theta) = -ik\int_0^\infty p\,\mathrm{d}p J_0(qp)\exp[2i\delta(p)]\,. \tag{4}$$

In order to convert this formulation into one in terms of the charge density it is convenient to rewrite Eq. (3) in the form

$$\delta(p) = -(\hbar c)^{-1}\int_{-\infty}^\infty \mathrm{d}z\int(\mathrm{d}^2p')\int_{-\infty}^\infty \chi(\boldsymbol{p}'^2+z'^2)\,[(\boldsymbol{p}-\boldsymbol{p}')^2+(z-z')^2]^{\frac{1}{2}}\,\mathrm{d}z'\,, \tag{5}$$

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([3]) G. Parzen: *Phys. Rev.*, **80**, 261 (1950).
([4]) G. Molière: *Zeits. f. Naturfor.*, **2 A**, 133 (1947).

where $\chi(p'^2+z'^2)$ is the nuclear charge density defined so that its integral over all space gives $-Ze^2$. We can then write Eq. (4) as

$$f(\theta) = -ik^{1-2i\gamma}\int_0^\infty \mathrm{d}p\, p^{1-2i\gamma} J_0(qp) \exp\left[-2i\Delta(p)\right], \tag{6}$$

where $\gamma = Ze^2/\hbar c$, and

$$\Delta(p) = -2\pi/(\hbar c)\int_p^\infty \mathrm{d}p' \int_{-\infty}^{\infty} \ln\,(p'/p) p' \chi(p'^2+z'^2)\,\mathrm{d}z'\,. \tag{7}$$

If we now restrict $\chi$ to a finite radial extent $R$ so that it vanishes for $p'^2+z'^2 \geqslant R^2$, then the entire contribution to $\Delta(p)$ will come from the region $p \leqslant p' \leqslant R$, and $(s^2 = R^2 - p'^2)$

$$\Delta(p) = -2\pi/(\hbar c)\int_p^R \mathrm{d}p' \int_{-s}^{s} \ln\,(p'/p) p' \chi(p'^2+z'^2)\,\mathrm{d}z', \qquad p \leqslant R, \tag{8}$$

$$= 0\,, \qquad p \geqslant R,$$

or

$$f(\theta) = -ik^{1-2i\gamma}\int_0^R \mathrm{d}p p^{1-2i\gamma} J_0(qp) \exp\left[-2i\Delta(p)\right] - ik^{1-2i\gamma}\int_R^\infty \mathrm{d}p p^{1-2i\gamma} J_0(qp)\,. \tag{9}$$

The Mott point-charge amplitude may now be separated out to give

$$\begin{aligned} f(\theta) = {} & (2k/q)^{1-2i\gamma}(\gamma/q)\Gamma(1-i\gamma)/\Gamma(1+i\gamma) + \\ & + i(kR)^{1-2i\gamma}[R/(2-2i\gamma)]_1F_2(1-i\gamma;\,1,\,2-i\gamma;\,-q^2R^2/4) - \\ & - ik^{1-2i\gamma}\int_0^R \mathrm{d}p p^{1-2i\gamma} J_0(qp) \exp\left[-2i\Delta(p)\right]. \end{aligned} \tag{10}$$

where $\Gamma(x)$ is the gamma function and ${}_1F_2$ is a generalized hypergeometric series.

In Eq. (10) we have achieved the desired separation of the amplitude into three components. Näively we may picture the second term of Eq. (10) as describing the effect of having « chopped out » a spherical cavity of radius $R$ from the overall point charge distribution associated with the first term, while the third term describes the effect of having « filled up » the cavity with the given charge distribution.

## 2. – Application to a uniform distribution.

As a simple illustration of the use of Eq. (10) in a practical problem, we consider the example of a uniform nuclear charge density. It must be kept in mind, however, that this choice. although it possesses the virtue of simplicity, is not too desirable since it violates the assumption of a slowly varying potential implicit in the use of a formula such as Eq. (2).

For a uniform distribution, Eq. (8) can be integrated directly to yield

$$(11)\quad \Delta(p) = -\gamma\left\{\frac{1}{3}[1-p^2/R^2]^{\frac{3}{2}} + [1-p^2/R^2]^{\frac{1}{2}} + \frac{1}{2}\ln\left[\frac{1-(1-p^2/R^2)^{\frac{1}{2}}}{1+(1-p^2/R^2)^{\frac{1}{2}}}\right]\right\},\quad p \leqslant R,$$

$$= 0 \qquad\qquad p \geqslant R.$$

This expression can be further simplified by noting that we are only concerned with small angle scattering (in other words, large impact parameters) so that we may reasonably expect the majority of the contribution to $\Delta(p)$ to come from regions close to $p = R$. In this case we can make the expansion

$$\ln[1+(1-p^2/R^2)^{\frac{1}{2}}] = (1-p^2/R^2)^{\frac{1}{2}} - \tfrac{1}{2}(1-p^2/R^2) + \tfrac{1}{3}(1-p^2/R^2)^{\frac{3}{2}} + \ldots,$$

truncating the series after the third term; and Eq. (10) becomes

$$(12)\quad f(\theta) = (2k/q)^{1-2i\gamma}(\gamma/q)\Gamma(1-i\gamma)/\Gamma(1+i\gamma) +$$
$$+ i(kR)^{1-2i\gamma}[R/(2-2i\gamma)]\,{}_1F_2(1-i\gamma;\,1,\,2-i\gamma;\,-q^2R^2/4) +$$
$$+ \{-i(kR)^{1-2i\gamma}(R/2\gamma)[U_1(2\gamma, qR) + iU_2(2\gamma, qR)] \equiv f_{ch}(\theta)\},$$

where $U_1$ and $U_2$ are Lommel functions of two variables.

If we multiply $f_{ch}(\theta)$ in Eq. (12) by its complex conjugate, we find

$$|f_{ch}(\theta)|^2 = [k^2R^2/(2\gamma)^2][U_1^2(2\gamma, qR) + U_2^2(2\gamma, qR)];$$

however, this is just proportional to the expression for the diffraction of spherical waves of wavelength $2\pi k^{-1}$ by a circular aperture of radius $R$. Thus it can be seen that the effect of the charge dependent contribution is essentially that expected from diffracted De-Broglie waves. This means that the charge distribution manifests itself primarily at the edges of the nuclear sphere, a result entirely consistent with the original hypothesis of small-angle, large impact-parameter scattering.

Although the above type of analysis is of value in that it lends some physical insight into the behavior of the charge dependent term, the diffraction

effects would be unobservable for this particular type of scattering, being cancelled by the same effects in the finite cut-off term. To see this, we note that the approximation made in arriving at Eq. (12) is equivalent to saying that $\Delta(p) \approx 0$ and $(p/R)^{-2i\gamma} \approx \exp[i\gamma(1 - p^2/R^2)]$. Under such conditions, the first and second terms of Eq. (9) have the same integrand so that the scattered amplitude just becomes that for a point charge, corresponding to the cancellation of the charge-dependent and finite-cut-off terms.

In conclusion, it should be emphasized that the above results have been derived by use of an expression, Eq. (2), which is valid only under certain very restrictive conditions [1] so that the range of application cannot be expected to be very broad; however, for the class of problems for which it is suitable, Eq. (10) has the advantage of displaying quite clearly the dependence of the amplitude on the parameters of the nuclear charge distribution, an advantage often lost in formulae having a wider range of applicability.

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The author would like to express his thanks to Dr. B. J. Malenka for many discussions concerning the details of these calculations and the use of the WKB type approximation in general.

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## RIASSUNTO (*)

Ricorrendo ad un'estensione relativistica della formula di Molière per l'ampiezza di scattering dell'elettrone al caso particolare di potenziali generati da distribuzioni di carica radiali, di estensione finita, si dimostra che l'ampiezza può essere suddivisa in tre contributi distinti: 1) l'ampiezza di scattering della carica puntiforme di Mott; 2) una correzione a questa dovuta al cut-off radiale finito, e 3) un termine caratteristico della distribuzione della carica. L'espressione risultante si applica al caso di una distribuzione di carica uniforme, per la quale si indica un'analogia ottica.

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(*) *Traduzione a cura della Redazione.*

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# Analysis of Inelastic Interactions of $K^+$-Mesons with Emulsion Nuclei (40÷150 MeV).

M. Grilli, L. Guerriero, M. Merlin and G. A. Salandin

*Istituto di Fisica dell'Università - Padova*
*Istituto Nazionale di Fisica Nucleare - Sezione di Padova*

(ricevuto il 9 Giugno 1958)

**Summary.** — 212.5 m of $K^+$ track have been followed in an unbiassed way in the energy interval from 40 to 150 MeV. The events found have been classified as follows: 1) 8 K-H interactions; 2) 184 inelastic scatterings (excluding the charge exchange); 3) 23 charge exchange. All these events have been divided into four energy intervals (see Table IV). In the present work, data on the $K^+$-nucleus and $K^+$-nucleon inelastic interactions have been given. For the $K^+$-nucleus interactions we give: (i) The variation of the total cross-section with the primary energy; (ii) The angular and energy loss distributions and their correlation; (iii) Further support for a nuclear repulsive potential. Starting from the inelastic events, an analysis has been made to find the elementary K-nucleon cross sections. Table VI summarizes the procedure followed in this analysis. The peculiar feature of the analysis consists in taking into account the correlation between the angular and energy loss distributions. We have chosen with a maximum likelihood method the angular distribution in the centre of mass system which reproduces the observed distributions for inelastic events. Particular care was taken to ensure that the separation between elastic and inelastic events would not alter the essential results of the analysis. We have also made some tests on the internal consistence of our analysis. The total and angular K-free nucleon and K-free neutron cross-sections (for scattering as well as for charge exchange) have been deduced with this analysis. Our values for elementary cross-sections have been compared with those estimated by other authors. Finally, we have evaluated the scattering amplitudes and the relative weight of the $T=1$ and $T=0$ states.

## Introduction.

In the past two years data have been collected in several American and European Laboratories [1-15] on the subject of $K^+$ interactions (for energy less than 200 MeV). This information has led to a rather well defined statement of the main characteristics of the $K^+$-interaction with matter.

It is useful in the presentation of these data to separate our knowledge of the $K^+$-nucleus interaction and the $K^+$-nucleon elementary cross-section.

For the former, a number of results already obtained were at hand (repulsive nuclear potential, total and angular inelastic cross-section).

An improvement of the above results relied prominently on an increase of the statistics and on a more accurate separation between elastic and inelastic events.

As to the $K^+$-nucleon interaction, a less definite situation resulted in a more complex and indirect solution.

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(1) J. E. LANNUTTI, W. CHUPP, G. GOLDHABER, E. HELMY, E. ILOFF, E. PEVSNER and R. RITSON: *Phys. Rev.*, **101**, 1617 (1956).

(2) N. N. BISWAS, L. CECCARELLI-FABBRICHESI, M. CECCARELLI, K. GOTTSTEIN, N. C. VARSHNEYA and P. WALOSCHEK: *Nuovo Cimento*, **5**, 123 (1957).

(3) G. COCCONI, G. PUPPI, G. QUARENI and A. STANGHELLINI: *Nuovo Cimento*, **5**, 172 (1957).

(4) M. BALDO-CEOLIN, M. CRESTI, N. DALLAPORTA, M. GRILLI, L. GUERRIERO, M. MERLIN, G. A. SALANDIN and G. ZAGO: *Nuovo Cimento*, **5**, 393 (1957).

(5) B. BHOWMIK, D. EVANS, S. NILSSON, D. J. PROWSE, F. ANDERSON, D. KEEFE, A. KENNAN and J. LOSTY: *Nuovo Cimento*, **6**, 440 (1957).

(6) M. WIDGOFF, A. PEVSNER, D. FOURNET-DAVIS, D. M. RITSON, R. SCHLUTER and V. P. HENRY: *Phys. Rev.*, **107**, 1430 (1957).

(7) T. F. HOANG, M. F. KAPLON and R. CESTER: *Phys. Rev.*, **107**, 1698 (1957).

(8) D. FOURNET-DAVIS: *Phys. Rev.*, **106**, 816 (1957).

(9) C. MARCHI, G. QUARENI, A. VIGNUDELLI, G. DASCOLA and S. MORA: *Nuov Cimento*, **6**, 1790 (1957).

(10) B. SECHI-ZORN and G. T. ZORN: *Phys. Rev.*, **108**, 1098 (1957).

(11) J. E. LANNUTTI, S. GOLDHABER, G. GOLDHABER, W. W. CHUPP, S. GIAMBUZZI, C. MARCHI, G. QUARENI, and A. WATAGHIN: *Reports of the Padua-Venice Conference* (September 1957), Session III, page 1.

(12) M. GRILLI, L. GUERRIERO, M. MERLIN and G. A. SALANDIN: *Reports of the Padua-Venice Conference* (September 1957), Session III, page 16.

(13) N. N. BISWAS, M. CECCARELLI and N. SCHMITZ: *Reports of the Padua-Venice Conference* (September 1957), Session III, page 25.

(14) L. T. KERTH, T. F. KYCIA and L. VAN ROSSUM: *Reports of the Padua-Venice Conference* (September 1957), Session III, page 28.

(15) G. IGO, D. G. RAVENHALL, J. J. TIEMANN, J. E. LANNUTTI, G. GOLDHABER, S. GOLDHABER and R. M. THALER: *Reports of the Padua-Venice Conference* (September 1957), Session III, page 32.

Indeed, only the Hydrogen collisions yielded straightforward information about the elementary cross-section (namely the $K^+$-p cross-section) whereas the details on $K^+$-n interaction could only be obtained indirectly from a model-dependent analysis of the $K^+$-nucleus encounters. The results of such an analysis depend critically on the manifold corrections applied to go from the experimental figures to the features of the elementary interaction.

The aim of the present paper is to enrich the statistics at energies ranging from 40 to 150 MeV and to improve the analysis of $K^+$-nucleus events in order to obtain more reliable conclusions on the total cross-section, and the angular cross-section of the $K^+$-nucleon scattering.

To improve the significance of data, the results of the present analysis have been applied (throughout the paper) to the bulk of our recent measurement and those already published by the Authors and other colleagues of the Padua plate-group [4].

TABLE I.

| Stack | Units | KI | KII | KIII |
|---|---|---|---|---|
| Dimensions (*) | $cm^3$ | $27\times39\times2.4$ | $20\times25\times6.0$ | $7.5\times15.0\times6.0$ |
| No. of plates (600 μm) (*) | — | 40 | 100 | 100 |
| Momentum of $K^+$ at entrance (*) | MeV/c | 340-370 | 280-460 | 420 |
| Time of flight from the target to the stack | $10^{-8}$ s | $\sim 1.3$ | $\sim 1.3$ | $\sim 2.4$ |
| Remarks | | Single focusing | Single focusing | Double focusing |

(*) The quoted figures refer to the part of the stacks available in our laboratory.

## 1. – Experimental data.

1·1. *Scanning procedure.* – The scanning was made along the track. Each track to be followed was selected with our usual criteria previously described (grain-count, beam collimation etc.). Further details on the selection criteria, contamination estimate, and confidence limits, involved in the KI and KII-stack scanning can be found in the paper previously quoted [4]. In the KIII-stack all these preliminary determinations were made easier because of the double-focusing of the $K^+$ beam.

1·2. *Selection of events.* – For the criteria adopted for the selection of events refer to the quoted paper (4). The only relevant difference in scanning the KIII-stack was to lower the projected acceptance angle for scattering events from 12° to 5°.

Table II summarizes the events observed and the track-path length scanned as grouped in four energy intervals ranging from 40 to 150 MeV.

TABLE II.

| $E_{(MeV)}$ | Meters | K-H | $K_0$ (> 20°) | $K_n$ | $O_0$ | $O_n$ | Df | Not analizable events |
|---|---|---|---|---|---|---|---|---|
| 40-60 | 30.5 | 1 | 67 | 2 | 5 | — | — | 3 |
| 60-80 | 44.5 | 1 | 83 | 7 | 3 | 2 | — | 6 |
| 80-110 | 62.5 | 2 | 94 | 30 | 5 | 7 | — | 6 |
| 110-150 | 75.0 | 4 | 66 | 43 | 6 | 6 | — | 1 |
| Total | 212.5 | 8 | 310 | 82 | 19+(6) (*) | 15 | 86 | 16 |

K-H Interaction of a $K^+$ with an Hydrogen nucleus;
$K_0$ Scattering of a $K^+$ with no additional prongs (excluding recoils);
$K_n$ Scattering of a $K^+$ with $n$ additional prongs ( » » );
$O_n$ Apparent charge-exchange of a $K^+$ with $n$ prongs ( » » );
$O_0$ Apparent charge-exchange of a $K^+$ with no prongs (stop) ( » » );
Df Decay in flight of a $K^+$.
(*) The six events $O_0$ occur within 30 μm from the air or glass surface. We have disregarded these events and correspondingly 10 % of the track length to compute the m.f.p. for this type of events.

1·3. *Analysis of the events.*

K-Hydrogen collisions. Eight interactions with free protons have been observed; the details are collected in Table III. From our data the total cross-section for the scattering K-p results to be $(12 \pm 4)$ mb as an average over the whole energy range considered. With the cut off angle in the la-

TABLE III.

| Event | $E_1$ (MeV) | $\vartheta_{C.M.}$ |
|---|---|---|
| $Pd_3$ | 47 | 112° |
| $Pd_4$ | 67 | 58° |
| $Pd_5$ | 85 | 96° |
| $Pd_6$ | 91 | 118° |
| $Pd_7$ | 126 | 118° |
| $Pd_8$ | 136 | 49° |
| $Pd_9$ | 135 | 26° |
| $Pd_{10}$ | 150 | 41° |

boratory system ($\geqslant 15°$) our estimate is in agreement with the accepted value of $(14.5 \pm 2.2)$ mb. ([11]).

Elastic and inelastic scatterings. By using the much greater statistics, we have improved the criterion for discriminating between elastic scatterings and inelastics ones as outlined in (4).

It is well known that the ratio $\sigma_E/\sigma_I$ of the « elastic » cross-section to the « inelastic », depends rather strongly on the scattering angle $\theta_{\text{Lab}}$ and on the energy of the primary K-meson. Therefore, taking advantage of the large number of events, we have subdivided our events into six intervals of $\theta_{\text{Lab}}$, for the same four intervals of energy as in Table II (see Fig. 1).

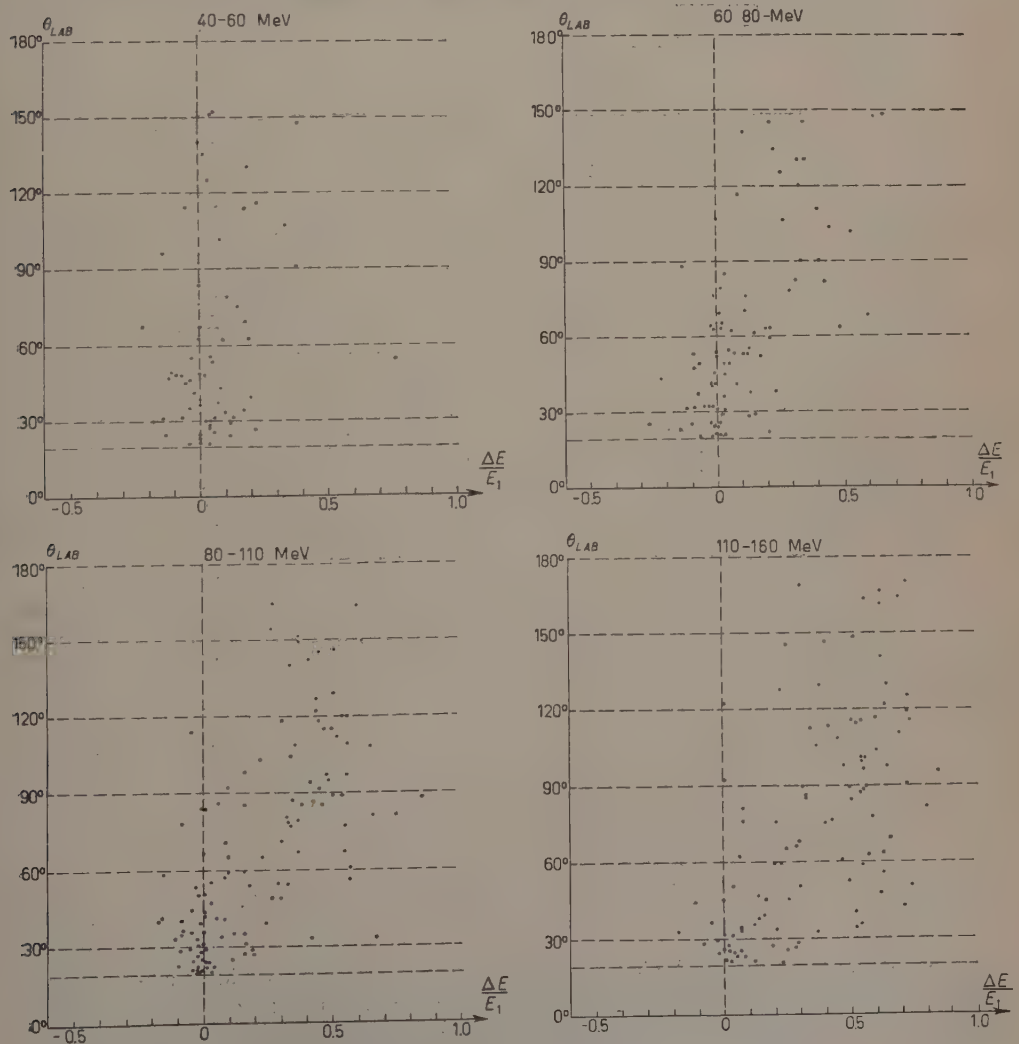


Fig. 1. – $\Delta E/E_1$ *vs.* $\theta_{\text{Lab}}$ distribution for all events.

To achieve a reasonable estimate of the contribution of inelastic scattering to the class of events termed as $K_0$, frequency diagrams have been plotted for the fractional energy loss $\Delta E/E_1$ (*) in such events.

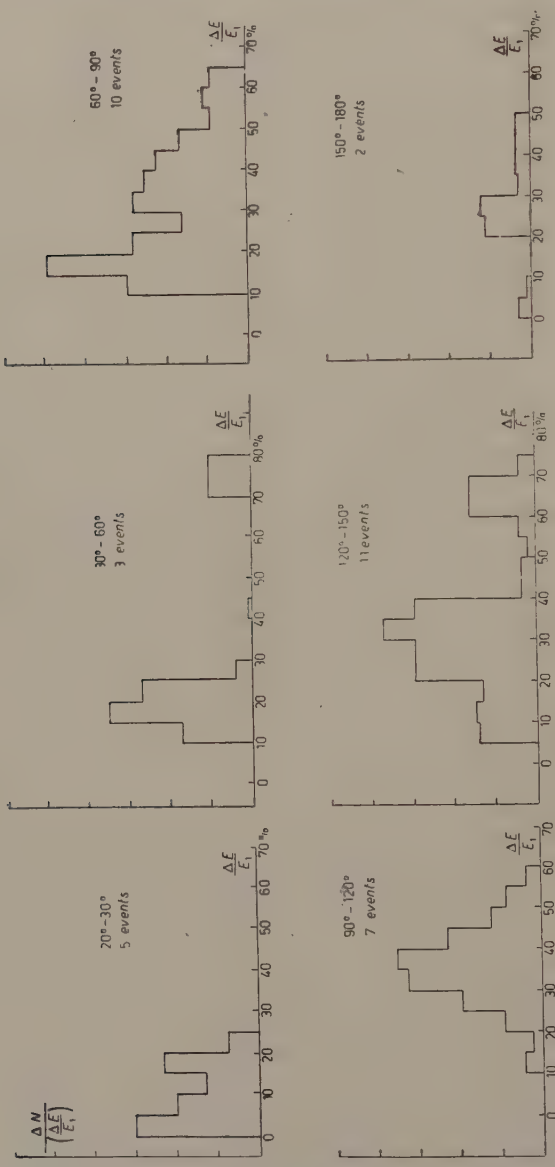


Fig. 2. – $\Delta E/E_1$ distribution for all inelastic events: *a*) $E_1$: (40÷80) MeV.

(*) For the estimate of $\Delta E/E_1$, for the events in K III stack, we have used a method which is slightly different from that described in (4). The energy of incoming K's ($E_1$) has been estimated by grain counting after careful calibration on stopping K's, and the energy of emerging K's ($E_2$) by range measurements.

After application of the usual subtraction method ([4]) and the addition of the events with associated prongs (which doubtlessly belong to the inelastic

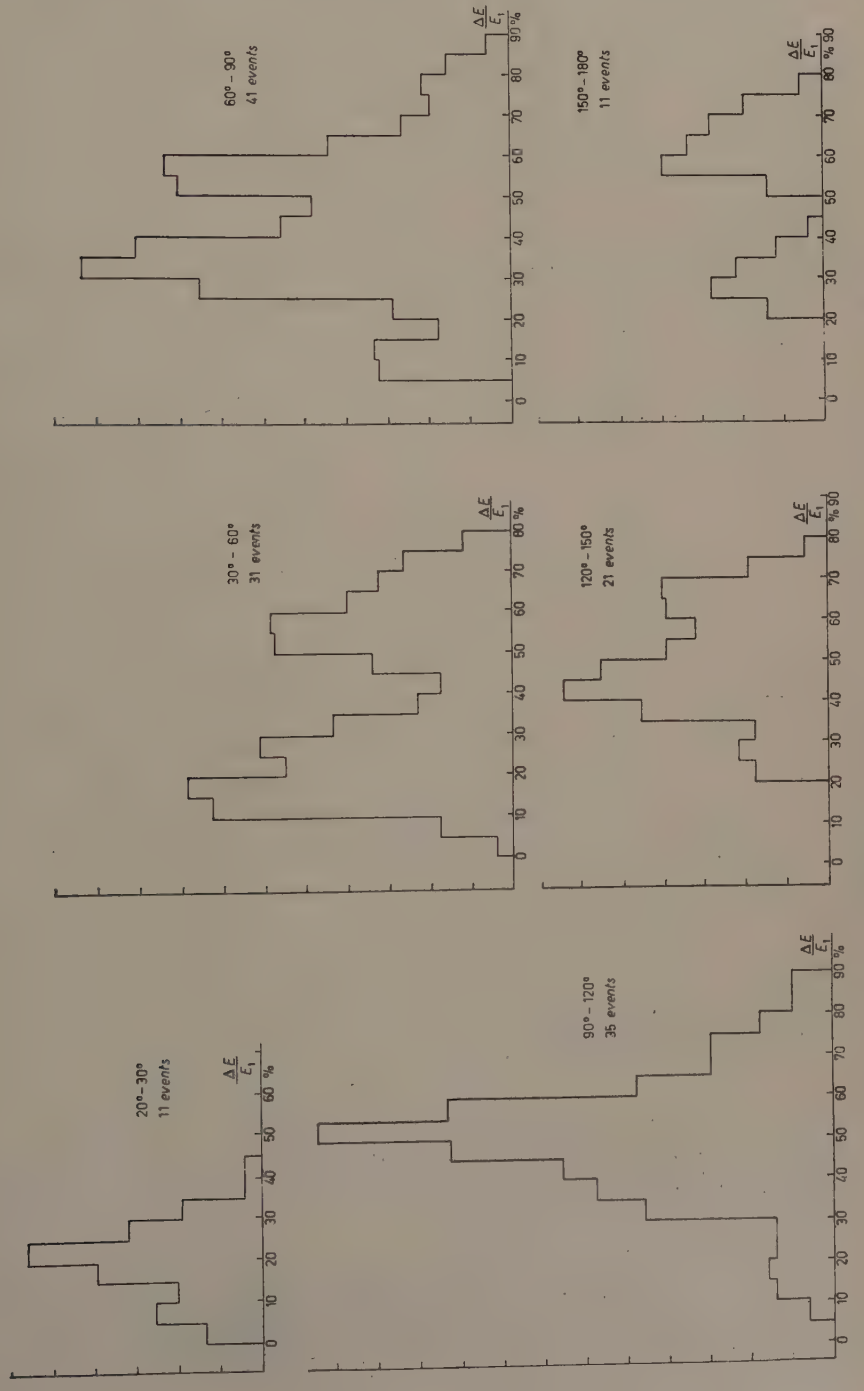


Fig. 2. – $\Delta E/E_1$ distribution for all inelastic events: *b*) $E_1$: (80 ÷ 150) MeV.

counterparts) we obtain the $\Delta E/E_1$ distribution for inelastic events, as shown in Fig. 2.

We would like to stress that any elastic-inelastic separation based on a subtraction method yields poor accuracy in the number of estimated inelastic events when most of the events are elastic, *i.e.* for low energy loss ($\Delta E/E_1 < 20\,\%$) and small angle of scattering ($\theta_{\text{Lab}} < 30°$).

As will be apparent from the following, this circumstance does not at all alter the results of our analysis, although some restrictions are to be made for its completeness.

Charge-exchanges. In all events with no emerging $K^+$, the nature of the primary K-meson has been checked by careful measurements of ionization *vs.* $\bar{\alpha}$. The nature of the secondary tracks has been determined by direct measurements, when possible. For prongs unsuitable for direct mass measurements, we have used a statistical method based on the fact that the range distribution for $K^+$ emerging from inelastic events is strongly different from that of prongs identified as not K's (Fig. 3).

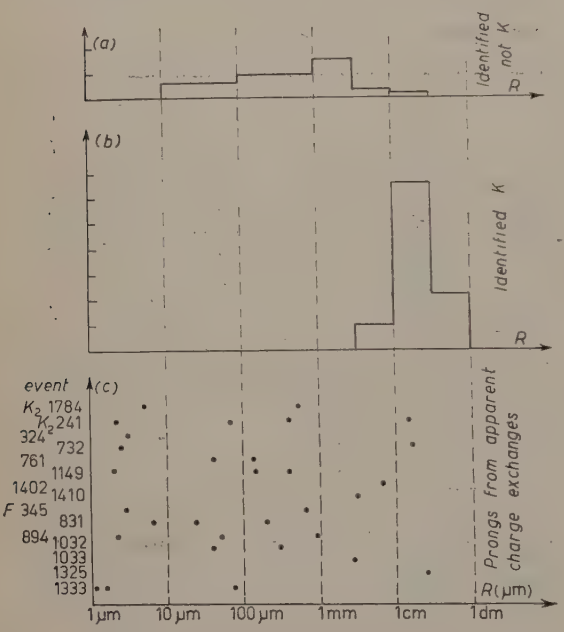


Fig. 3. – Range distribution of all interaction prongs: *a*, identified not K's; *b*, identified K's; *c*, unidentified.

The percentage of K's present among the unidentified prongs from apparent charge exchanges has been evaluated by comparison of their range distribution with that reported in Fig. 3*a*, *b*. We have taken into account in this evaluation also the visibility factor for a decay as a function of the depth in the emulsion of the end of the prongs.

To ascertain the amount of stops attributable to decays in flight (due to lack of detectable decay particles), the visibility factor for this type of events has been used (see [16]).

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([16]) B. Bhowmik, D. Evans, S. Nilsson, D. J. Prowse, F. Anderson, D. Keefe A. Kennan, N. N. Biswas, M. Ceccarelli, P. Waloschek, J. E. Hooper, M. Grilli and L. Guerriero: *Nuovo Cimento*, **5**, 994 (1957).

We have excluded all those stops lying within 30 μm from the air or glass surface (unprocessed emulsion).

1·4. *Results.* – The present paper deals with experimental results on inelastic scattering and charge-exchange scattering. The work on elastic scattering is still in progress.

*a*) Total cross-section. The total cross-sections for inelastic scattering and charge-exchange scattering are quoted in Table IV.

TABLE IV.

| $E$ (MeV) | Inelastic scattering | | | Charge exchange | | |
|---|---|---|---|---|---|---|
| | Events | | $\sigma$ (K-nucleus) (mb) (+) | Events | | $\sigma$ (K-nucleus) (mb) (+) |
| | $K_0$ (*) | $K_n$ | | $O_0$ | $O_n$ | |
| 40-60 | 7 | 2 | $63 \pm 13$ | 3 | 3 | $18 \pm 8$ |
| 60-80 | 20 | 7 | $132 \pm 18$ | | | |
| 80-110 | 31 | 30 | $210 \pm 22$ | 7 | 10 | $28 \pm 7$ |
| 110-150 | 44 | 43 | $250 \pm 21$ | | | |

(*) Corrected for geometrical loss.
(+) Quoted errors are statistical r.m.s.

The Fig. 4 shows the dependence on $E_1$ of the total cross-section for inelastic scattering and charge exchange. To the crude experimental figures

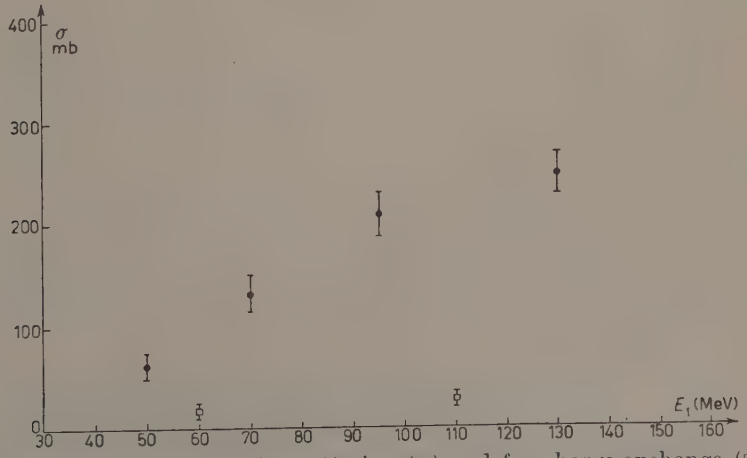


Fig. 4. – Cross-section for inelastic scattering (●) and for charge exchange (□) *vs.* $E_1$.

a correction has been applied for geometrical loss due to the cut-off acceptance angle, as stated above in Sect. 1·2.

*b*) Angular cross-section of inelastic events. Fig. 5 shows the angular cross-section in the laboratory frame for inelastic events (after corrections for geometrical loss) for $E_1$ between 80 and 150 MeV. The distribution appears to be nearly isotropic, if allowance is made for those inelastic events, whose fractional energy loss does not exceed 20%. If such events are taken into account, a slight forward peaking then appears.

As the number of inelastic events with $\Delta E/E_1 < 20\%$ is not well estimable (see Sect. 1·3) we have carried out an analysis which does not require the knowledge of this number.

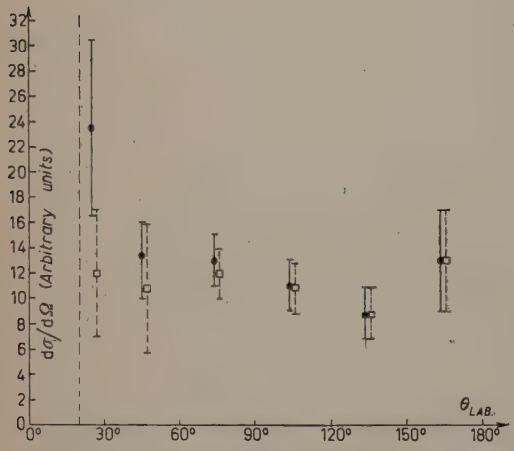


Fig. 5. – Angular distribution (lab. system) for inelastic events for the energy interval (80÷150) MeV. ● All inelastic events; □ Only inelastic events with $\Delta E/E_1 > 20\%$.

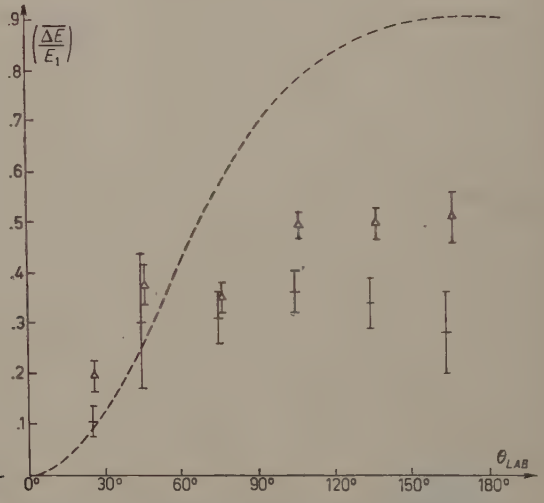


Fig. 6. – Average energy loss $\overline{\Delta E/E_1}$ *vs.* scattering angle $\theta_{\text{Lab}}$ (the errors reported in this and in the next figure are that of the mean value): + (40÷80) MeV; △ (80÷150) MeV. Broken line: K-free nucleon collision.

*c*) Correlations of the fractional energy loss $\Delta E/E_1$ to the angle of scattering $\theta_{\text{Lab}}$ and to primary energy $E_1$. The distributions of Fig. 1 show the stochastic dependence of $\Delta E/E_1$ from the angle of scattering in the laboratory system $\theta_{\text{Lab}}$ as well as from the kinetic energy $E_1$ of the interacting $K^+$.

To characterize in a simple way the correlations appearing from the foregoing diagrams, we have plotted the mean values of the parameter $\Delta E/E_1$ versus the angle $\theta_{\text{Lab}}$ and versus the energy $E_1$ resepctively (Fig. 6, 7).

As it appears from Fig. 6 the experimental $(\overline{\Delta E/E_1})$ shows a dependence on $\theta_{Lab}$, but less marked than expected for a K⁺-free nucleon collision (dotted line). For the low energy interval this dependence appears to be even less marked).

The data quoted in Fig. 7 show a clear trend of $(\overline{\Delta E/E_1})$ towards higher values as the energy $E_1$ is increased.

These data supply further evidence for a nuclear repulsive potential K-nucleus, as discussed in (4) and (2,3,5,12).

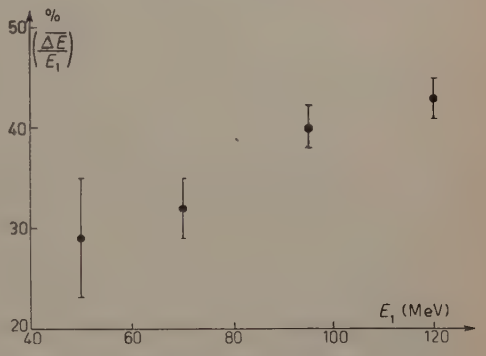


Fig. 7. – Average energy loss $\overline{\Delta E/E_1}$ *vs.* $E_1$ distribution.

## 2. – Discussion of experimental results.

The following sections are devoted to an analysis of the data related to inelastic events.

2·1. *Total cross-sections* K⁺*-nucleus.* – To evaluate the mean free paths in nuclear matter from the experimental cross-section quoted in Table IV, corrections should be applied to account for the effect of the Coulomb repulsive field and for the nuclear field acting on the K⁺. For the former effect, the correction factor is:

$$\left(1 - \frac{V_c}{E_1}\right)^{-1},$$

where $V_c$ is the Coulomb potential at the boundary of the nucleus ($V_c \sim 10$ MeV) and $E_1$ the kinetic energy of the K⁺ (17).

A further correction due to the nuclear contribution of the repulsive field, has been evaluated numerically by calculating the average shortening of the K⁺-meson path inside nuclear matter because of the refraction at the nuclear boundary, For such calculations a square potential well 15 MeV deep has been considered as the best estimate from the data of various investigators (3-5,7-9,12,14,15).

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(17) J. M. Blatt, V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), page 346.

After such corrections the experimental cross-sections yielded the mean free paths in nuclear matter at various energies as follows:

| $E_1$ (primary energy | MeV | $40 \div 80$ | $80 \div 110$ | $110 \div 150$ |
|---|---|---|---|---|
| $\Lambda$ (experimental m.f.p.) | cm | $178 \pm 28$ | $91 \pm 11$ | $78 \pm 7$ |
| $\lambda$ (m.f.p. in nuclear matter) | fermis | 30 | 13 | 11 |

The values of $\lambda$ have been evaluated using the relation reported by BRUECKNER *et al.* ([18]), calculated for the standard nuclear emulsion.

2·2. *Elementary cross-sections.* – In order to obtain the angular cross-section of the $K^+$-free nucleon scattering from the experimental data, we have sought the cross-section in the center of mass system which reproduced the observed distribution of the scattering angles, of the energy loss and their correlation.

Our analysis rests on two hypothesis:

*a*) Each inelastic event is due to a single collision with one nucleon inside the nucleus. Since the probability of a $K^+$ undergoing two or more such collisions inside the same nucleus is very small (less than 3% for $E_1 = 100$ MeV, assuming $(\overline{\Delta E}/E_1) = 40\%$ in the first collision) the hypothesis appears quite reasonable.

*b*) Concerning the dynamics of the collisions the $K^+$ and the nucleons are considered as being free in a rectangular potential well of suitable depth.

Hence we have adopted a Fermi-model with a $P_{max} = 241$ MeV/c, corresponding to a nuclear radius $R = 1.25 \cdot A^{\frac{1}{3}} \cdot 10^{-13}$ cm ([19]).

A nuclear potential K-nucleus of 15 MeV has been assumed in addition to the Coulomb one. We shall discuss at length the influence of a possible variation of this value on the conclusions concerning the K-nucleon cross-section.

For a given scattering angle in the center of mass system, we have calculated the angle and energy loss distributions to be expected in the laboratory system. Such an evaluation has been performed by taking into account the conservation laws in the elementary collisions as well as a complete set of

([18]) K. A. BRUECKNER, R. SERBER and K. M. WATSON: *Phys. Rev.*, **84**, 258 (1951).
([19]) J. M. C. SCOTT: *Progress Nucl. Phys.*, **5**, 157 (1956).

possible values for the unobserved parameters, *e.g.* nuclear momentum, azimuthal scattering angle etc.

Of course only collisions allowed by the exclusion principle have been considered.

In this way it was possible to predict the $d\sigma/d\Omega$ and $\Delta E/E_1$ distribution in the laboratory system, corresponding to any angular cross-section in the center of mass system assumed for the K+-free nucleon collision.

In our calculations refraction at the entrance of the nucleus has been neglected since our analysis is restricted to the class of events having $E_1 \geqslant 80$ MeV. At this energy the refraction is less than 10°: moreover a balancing effect is to be expected which reduces the overall average shift to a great extent. Table V (A, B) shows an arrangement of the experimental data according to the following criteria:

1) Events with $\Delta E/E_1 < 20\%$ have been disregarded because of the possible contamination of elastic scatterings. A check of external consistency for the analysis based on this truncated distribution gave satisfactory results (*).

Moreover the expected numbers of such events would depend strongly on the maximum momentum of the nucleons, via the Pauli principle.

2) The grouping of events in wide angular intervals allows the deductions drawn from the relative number of events in each block of the table to be nearly independent of the refraction effect at the exit of the K+ from the nucleus. Indeed the refraction angle to be expected at the exit may be much larger than that at the entrance if the K+ undergoes a strong energy loss by interaction inside the nucleus.

3) The experimental accuracy in the $\Delta E/E_1$ measurements ($\sim 5\%$) would allow intervals narrower than used by us (*e.g.* 10%). On the other hand, the expected $\Delta E/E_1$ distribution can be evaluated since the nuclear potential is rather well known. We have used 20% intervals in order not to reduce too much the number of events in each block.

In some preceding works [3-5], evidence was found of a not isotropic distribution in the center of mass system. Our first aim in this paper, therefore, is to use better statistics and an improved analysis for learning whether an isotropic distribution is completely ruled out by the experimental data.

Assuming that

(*a*) $$(d\sigma/d\Omega)_{CM} = a + b \cos \vartheta_{CM} ,$$

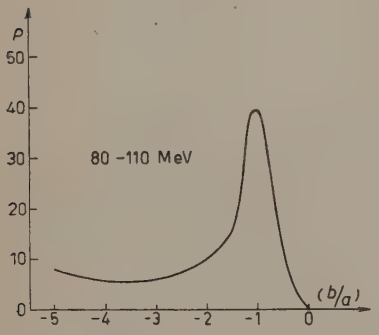
(*) The angular cross-section deduced by our analysis leads to forecast 14% of inelastic events with $\Delta E/E_1 < 20\%$. Actually 17 events were found, which constitute 12% of the total.

we have calculated for various values of $b/a$ the corresponding distribution of the angle $\vartheta_{\text{Lab}}$ and of $\Delta E/E_1$, normalized to the number of our experimental events.

By the usual method of $\chi^2$ we have calculated the probability that the experimental distribution derived from a $(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\text{CM}}$ of the form $(a)$ for various values of the parameter $(b/a)$. The plot of the results of this probability *vs.* $b/a$ is shown in Fig. 8. These results are similar for both energy intervals. Hence we have added the data for the two energy intervals and again applied the $\chi^2$ method. Fig. 9*b* shows the plot of the probability valid for the whole interval $(80 \div 150)$ MeV. From this figure it appears that an isotropic cross-section is to be excluded; the value $b/a$, which gives a maximum probability for obtaining the experimental distributions turns out to be $b \simeq -\frac{5}{8}a$.

It seems interesting to check whether such anisotropy is due to any assumptions made in the course of the analysis.

Assuming the two general hypotheses mentioned above (*a*, *b*, p. 216) and considering the effects due to refraction and double scatterings as negligible it is important to verify how our conclusions depend on the value of the potential: $V_c + V_n$. Therefore we have re-calculated the above probability distributions for two extreme values, *viz.*, $V_c + V_n = 15$ MeV and $V_c + V_n = 35$ MeV. Fig. 9*a*, *c* show these distributions. We see that the best fit of experimental data is always a backward peaked cross-section $(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\text{CM}}$.



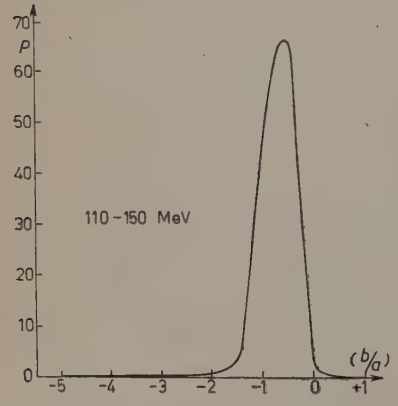




Fig. 8. – Probability arising from the $\chi^2$-test for the value of the ratio $b/a$ (see text).

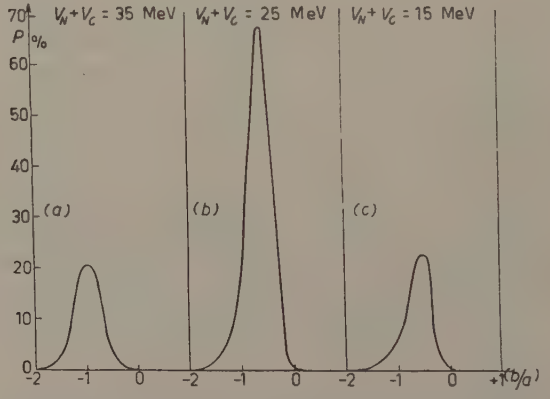



Fig. 9. – *a*) *b*) *c*) Probability curves for three values of the nuclear repulsive potential $V_n = 25,\ 15,\ 5$ MeV.

By comparing the three distributions of Fig. 9 *a*, *b*, *c*, it is clear that our experimental data strongly favour a value of $V_c + V_n = 25$ MeV. This value of 25 MeV is in agreement with that obtained by us [4,12] and other authors [2,3,5,7-9,12,14,15]., who used different methods.

We now proceed with our analysis in an effort to distinguish the separate contributions due to protons and neutrons. On the basis of present data for $K^+$-H events [11], we have assumed that the $K^+$-nucleon interaction in the state $T=1$, is practically of *S*-wave type. Hence the anisotropy which we have found for $(d\sigma/d\Omega)_{CM}$ for $K^+$-free nucleon is due to $K^+$-neutron scattering in state $T=0$. On this hypothesis starting from the approximate relationships (1) and (2), of BALDO *et al.* [4] (*) the cross-section for K-nucleon scattering assumes this form:

$$(b) \qquad \frac{d\sigma}{d\Omega} = \lambda^2 \left[ a_{10}^2 + \frac{1}{4}\left\{ (a_{10} + a_{00}) + 3(a_{11} + a_{01}) \cos \theta_{CM} \right\}^2 \right].$$

This differs from the form (*a*) used above (page 217).

Formula (*b*) reduces to (*a*) if one subtracts from $(d\sigma/d\Omega)_{CM}$ the term $\lambda^2 \cdot a_{10}^2$ which arises from K-p scattering. Therefore if we subtract from the experimental distribution of K-nucleus events (of Table V, *a*, *b*) the expected number of K-bound proton collisions (as can be calculated from the knowledge of the elementary cross-section) we get the expected distribution of K-bound

TABLE V.

*a* — Experimental data, $80 \leq E_1 \leq 110$ MeV

| $\theta_{LAB}$ \ $\Delta E/E_1$ | 20–40 | 40–60 | 60–80 | 80–100 | % |
|---|---|---|---|---|---|
| 20–30 | 4 | 1 | | | |
| 30–60 | 5 | 2 | 1 | | |
| 60–90 | 8 | 6 | 2 | 2 | |
| 90–120 | 4 | 10 | 1 | | |
| 120–150 | 3 | 8 | 2 | | |
| 150–180 | 3 | | 1 | | |

*b* — $110 \leq E_1 \leq 150$ MeV

| $\theta_{LAB}$ \ $\Delta E/E_1$ | 20–40 | 40–60 | 60–80 | 80–100 | % |
|---|---|---|---|---|---|
| 20–30 | 5 | | | 1 | |
| 30–60 | 7 | 5 | 3 | | |
| 60–90 | 7 | 8 | 4 | | |
| 90–120 | 3 | 8 | 7 | 1 | |
| 120–150 | 3 | 2 | 4 | | |
| 150–180 | 1 | 1 | 4 | | |

*c* — K-Proton events ($80 \leq E_1 \leq 150$ MeV) (from $\sigma = 14.5$ mb; *S* wave)

| $\theta_{LAB}$ \ $\Delta E/E_1$ | 20–40 | 40–60 | 60–80 | 80–100 | % |
|---|---|---|---|---|---|
| 20–30 | 2.5 | 0.1 | 2.3 | | |
| 30–60 | 14.5 | 8.0 | | | |
| 60–90 | 11.7 | 11.3 | 8.9 | | |
| 90–120 | 6.1 | 7.7 | | | |
| 120–150 | 2.1 | 4.0 | 3.8 | | |
| 150–180 | 0.3 | 0.9 | | | |

*d* — K-Neutron events ($80 \leq E_1 \leq 150$ MeV)

| $\theta_{LAB}$ \ $\Delta E/E_1$ | 20–40 | 40–60 | 60–80 | 80–100 | % |
|---|---|---|---|---|---|
| 20–60 | 4.0 | 0.0 | 2.7 | | |
| 60–120 | 4.2 | 13.0 | 8.1 | | |
| 120–180 | 7.6 | 6.1 | 7.2 | | |

(*) We have changed in these relationships the sign of the $a_{Tl}$ for the $T=0$ state, according to the usual convention.

neutron scatterings. By repeating the analysis for these events alone, we can arrive at the cross-section $(d\sigma/d\Omega)_{CM}$ for K-free neutron.

The distribution of events due to K-bound proton scatterings in the nucleus has been derived by assuming an isotropic cross-section for K-H scattering with $\sigma_{tot.} = (14.5 \pm 2.2)$ mb ([11]). In making this calculation, we have considered fully the screening effect of the nucleus ([18]), as well as the Pauli exclusion principle.

Table V, *c*, *d* shows the distribution which we obtained from this calculation for K-proton and K-neutron collisions, respectively.

Repeating our analysis for neutrons alone, for a cross-section $(d\sigma/d\Omega)_{CM}$ of type (*a*), we have found the most probable value of the ratio $b/a$ to be $\leqslant -5$ (*).

To this figure, statistical errors are attached:

*a*) those arising from our experimental data;

*b*) those arising from the total and angular cross-section K-p;

*c*) fluctuations in the number of K-p collisions in the nucleus, that we have estimated from $\sigma_{Kp}$.

Hence, by taking into account these statistical fluctuations, we arrive at a value for $b/a < -1$ with a 90% confidence limit.

From the ratio $b/a$, *i.e.* from the angular distribution in center of mass system for K-neutron scattering, we can now evaluate the cut-off introduced by the Pauli principle. With this correction for the Pauli principle and assuming *e.g.* $-1 \geqslant b/a \geqslant -5$, the $\sigma_{\text{K-neutron (free)}}$ for scattering becomes $(5.4 \div 6.1)$ mb.

We can now extend the analysis by starting from $\sigma_{K\text{-}p}$ and $\sigma_{K\text{-}n}$ to calculate the scattering amplitudes $a_{10}$, $a_{00}$, $a_{01}$ ([4]) (+) for various values of the ratio $b/a$ relative to the neutron. For every value of this ratio we have found two sets of solutions (referred as *A* and *B*) for the amplitudes $a_{Tl}$.

We have tried to decide between the two possible solutions and arrive at a better value for $(b/a)_{neutron}$ by using the results of charge-exchange. In fact for every $b/a$ and for each solution (*A* and *B*) the expected charge-exchange cross-section can be calculated having made the usual corrections for Coulomb, Pauli and screening effects. Fig. 10,*a* shows the comparison between our calculations and experimental data (×). From this comparison

(*) Adding the data of B. Bhowmik *et al.* ([5]) relative to the same energy interval we found the same result for the ratio $(b/a)$.

(+) We have assumed that $a_{Tl}$ are real quantities. This approximation seems to be reasonable at our energies, as the cross-sections are rather small.

(×) We have assumed $\sigma_{CE} = (46 \pm 7)$ mb for K-nucleus interactions, as an average of the results now available ([4,5,11,12]) at $E_1 = 110$ MeV.

it can be seen that $b/a \sim -1.2$ and only solution $B$ has a physical meaning. Assuming therefore $b/a = 1.2$ we obtain see Fig. 10,*b*)

$a_{10} = -0.39 \pm 0.03$, $a_{00} = 0.0 \pm 0.07$,

$a_{11} = 0$ (hypothesis), $a_{01} = +0.16 \pm 0.05$.

These values have been calculated for a mean energy $E_1 = 110$ MeV.

The limiting values reported for the $a_{Tl}$ account for statistical errors and have been evaluated for $b/a = -1.2$. The total error would result by including the uncertainty in $b/a$ (see Fig. 10, *b*).

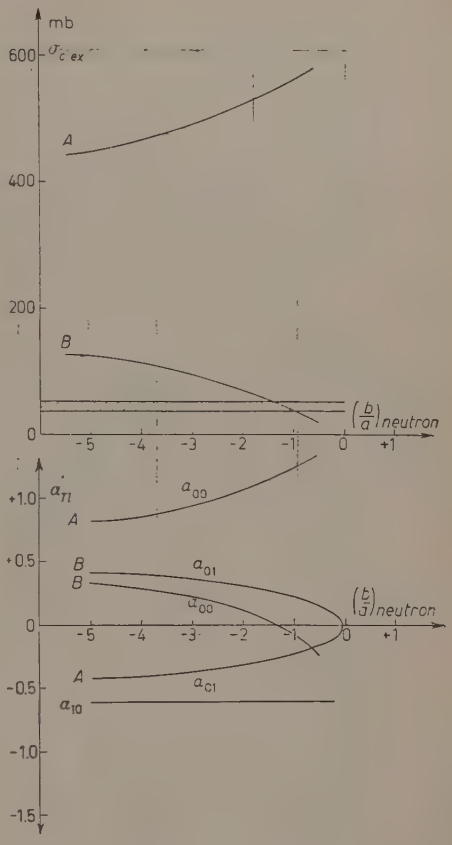



Fig. 10. – Charge-exchange K-nucleus cross-section (Fig. 10, *a*) and values of scattering amplitudes (Fig. 10, *b*) calculated as a function of $b/a$ (for neutrons) according to the solutions $A$ and $B$ (see text).

## 3. – Conclusions.

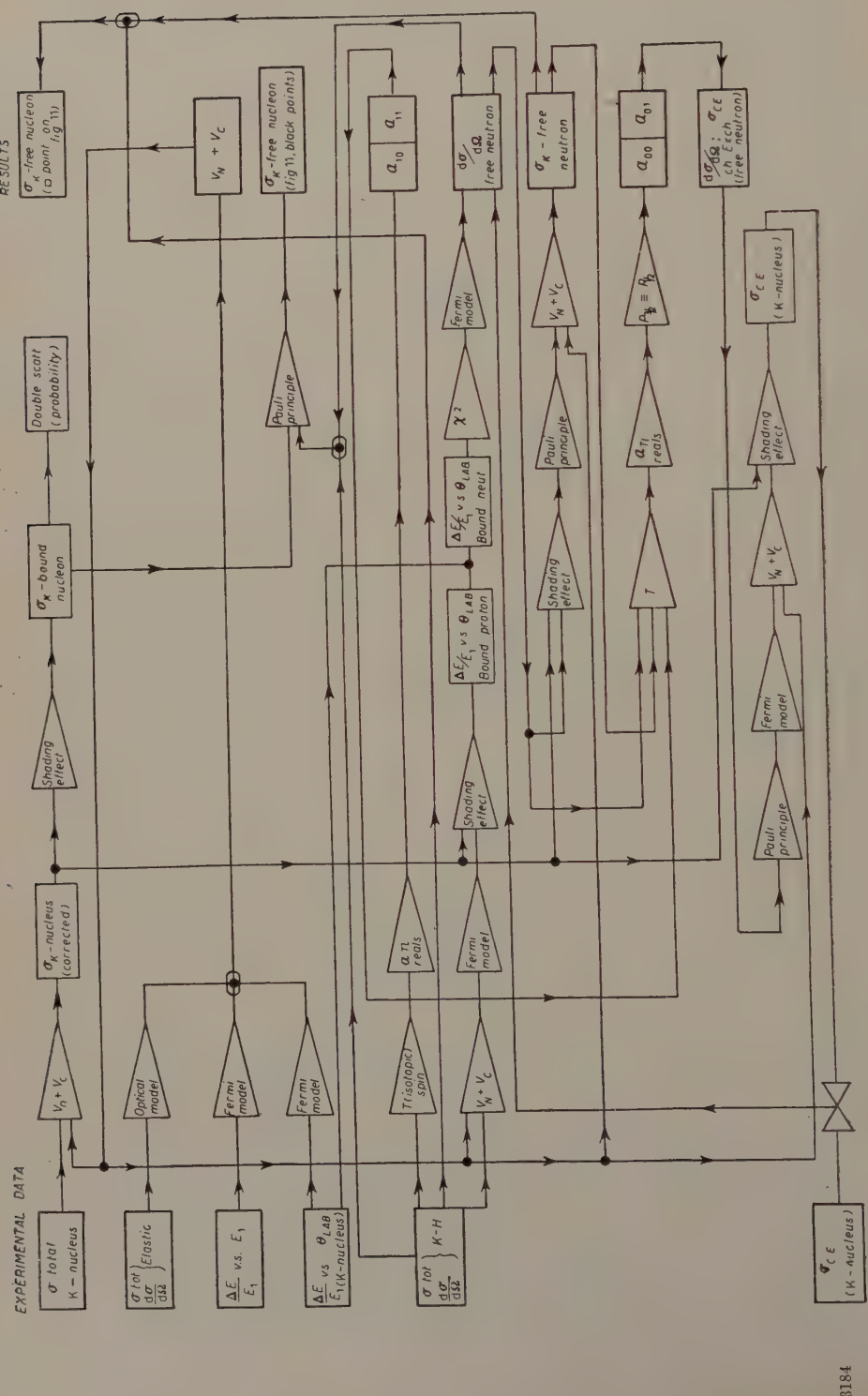
Table VI summarizes schematically the procedure used in our analysis to obtain the following results:

*K+-nucleus interactions.*

1) The total cross-section *vs.* primary energy $E_1$ is given in Fig. 4. The marked dependence on $E_1$ is reduced if the effect due to the potential $V_N + V_C$ is taken into account.

We note that the energy dependence that we found for the inelastic cross-section is in agreement with that found by Bhowmik *et al* ([5]), but is clearly different form that found by Lannutti *et al.* ([11]). These latter authors, however, have used a crude cut-off at $\Delta E/E = 10\%$ to discriminate between elastic and inelastic events. This difference in the criterion used for the discrimination « elastic-inelastic » leads to strong discrepancies for $E_1 < 100$ MeV.

TABLE VI.


EXPERIMENTAL DATA
RESULTS
σ total K – nucleus
σ tot dσ/dΩ Elastic
ΔE/E1 v.s. E1
ΔE/E1 vs θLAB (K-nucleus)
σ tot dσ/dΩ K-H
σc.e (K-nucleus)
Vn + Vc
Optical model
Fermi model
Fermi model
Trisotopic spin
Vn + Vc
σK – nucleus (corrected)
Shading effect
σK – bound nucleon
Double scatt (probability)
Pauli principle
σK – free nucleon (□ point fig 11)
VN + Vc
σK – free nucleon (fig 11, black points)
a10
a11
a Tl reals
Fermi model
Shading effect
ΔE/E1 vs θLAB Bound proton
dσ/dΩ free neutron
Fermi model
χ²
ΔE/E1 vs θLAB Bound neut
Shading effect
σK – free neutron
VN + Vc
Pauli principle
Shading effect
I
a00
a01
a Tl reals
PN ≡ PP
dσ/dΩ; σc.e Ch Exch (free neutron)
σc.e (K-nucleus)
Shading effect
VN + Vc
Fermi model
Pauli principle
3184

2) Fig. 5 shows the angular cross-section for $E_1$ between 80 and 150 MeV. The peak at small angles, turns out to be attributable to inelastic scatterings with small energy loss (see footnote page 217).

3) A nuclear repulsive potential $V_n \simeq 15$ MeV is confirmed (see Fig. 9).

4) Further evidence for the presence of such a repulsive potential is supplied by the trend of $\Delta E/E_1$ as a function of $E_1$ (Fig. 7 and GRILLI *et al.* ([12])).

K+-*nucleon interaction.*

1) The scattering cross-section per nucleon (charge exchange not included) derived from K-nucleus cross-sections after correction for Pauli principle and screening effect shows an energy dependence far less marked than the K+-nucleus cross-section (Fig. 11). Since the Pauli correction at low energy is strongly dependent on the choice of the momentum distribution inside the nucleus, no quantitative statement appears to be reliable about the energy dependence of $\sigma_{\text{K-nucleon}}$.

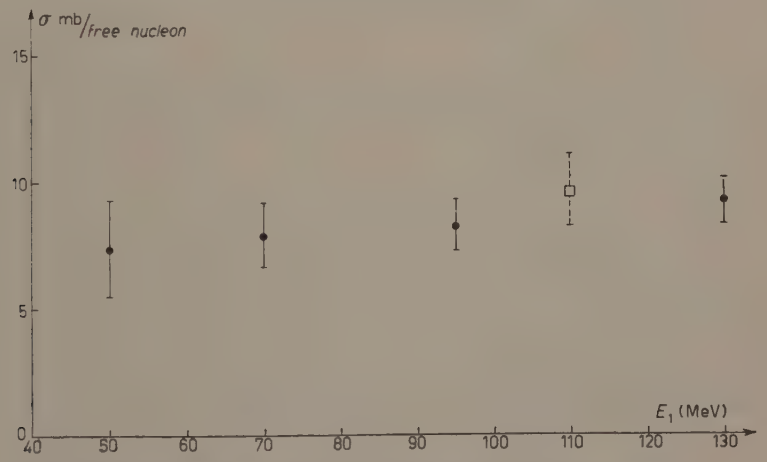



Fig. 11. – Cross-section for scattering K-nucleon *vs.* primary energy $E_1$. The value derived from our analysis for this cross-section has been reported as □.

2) As a conclusion of our analysis, we find that a scattering cross-section K+-free nucleon (at $E_1 = 110$ MeV)

$$(c) \qquad \sigma_{KN}(\theta_{CM}) = \frac{1}{4\pi}(8.9 - 4.7\cos\theta_{CM} + 2.3\cos^2\theta_{CM}) \text{ mb/sr nucleon},$$

gives the best fit to the experimental results. The comparison of the experi-

mental distributions relative to the scatterings, and the expected distributions (see Fig. 12) show satisfactory agreement.

The above $\sigma(\theta_{CM})$ is less peaked (forward and backward) than the cross-section given by COCCONI *et al.* [3]. The latter one, however, does not reproduce well the experimental distribution of $\Delta E/E_1$.

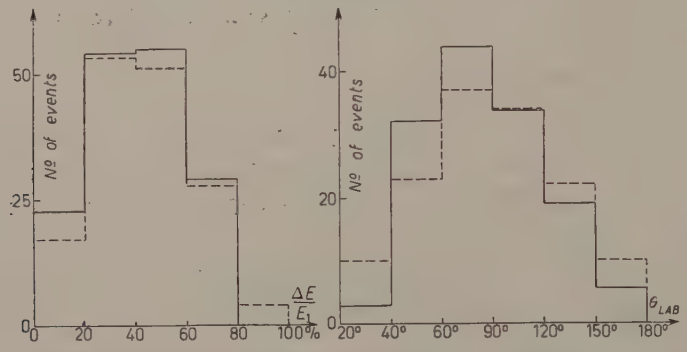



Fig. 12. – Comparison of the experimental distribution (dotted line) and that derived from the analysis (solid line).

In Fig. 11 we have reported the total cross-section corresponding to (*c*). There is a good agreement between the values obtained from K-nucleus cross-section and from the analysis.

$K^+$-*neutron scattering.* – The elementary cross-section for $K^+ + n = K^+ + n$ scattering results of the form

$$\sigma_{Kn}(\theta_{CM}) = (a + b \cos \theta_{CM})^2$$

with $b/a \sim -1.2$.

The total cross-section results at $\bar{E}_1 = 110$ MeV

$$\sigma_{Kn} = (5.7 \pm 1.9) \text{ mb} .$$

*Charge-exchange scattering.* – The total cross-section $\sigma_{CE}$ for charge-exchange on free neutron turns out to be $(E_1 = 110 \text{ MeV}) \sigma_{CE} = (5.7 \pm 1.9)$ mb.

The angular distribution appears to obey a law of the type:

$$(d) \qquad \sigma_{CE}(\theta_{CM}) = \frac{1}{4\pi} (3.7 + 9.4 \cos \theta_{CM} + 5.9 \cos^2 \theta_{CM}) \text{ mb/sr neutron} .$$

The Pauli correction for a cross-section of such a shape is rather strong ($\sim 50\%$). Furthermore, if corrections are applied to account for screening

effect, for Coulomb potential and for $(A - Z)/A$ ratio in the nuclei, the total charge exchange cross-section per bound nucleon turns out of the order of 1 mb. Hence one is able to explain the tiny number of charge-exchange events observed by experiment.

To compare the data now available on the $\sigma_{\text{K-neutron}}$ (scattering + charge-exchange) we collected these results in the following table:

TABLE VII.

| $E_1$ (MeV) | $(\sigma_{Kn})_{scatt.}$ (mb) | $\sigma_{CE}$ (mb) | $(\sigma_{Kn})_{total}$ (mb) | Authors |
|---|---|---|---|---|
| 40÷120 | | | $2 \pm 2$ | BHOWMIK *et al.* [5] |
| 60÷180 | $5.8 \pm 3.1$ | $4.0 \pm 0.8$ | $9.8 \pm 3.0$ | LANNUTTI *et al.* [11] |
| 80÷150 | $5.7 \pm 1.9$ | $5.7 \pm 1.9$ | $11.4 \pm 2.8$ | present work |
| 140÷220 | | | $10.3 \pm 3.5$ | ZORN *et al.* [10] (*) |
| 190 | | | $7.5 \pm 2.1$ | KERTH *et al.* [14] |

(*) The data of Zorn *et al.* have been derived from the $\sigma_{\text{K-nucleon}}$ given by these authors assuming $\sigma_{Kp} = (14.5 \pm 2.2)$ mb.

The results are in a rather good agreement and only the data of BHOWMIK *et al.* are slightly low.

*Relative weight of the isotopic spin state.* – The scattering amplitudes $a_{Tl}$ have been evaluated assuming: (i) that these quantities are real (see footnote (*) page 220); (ii) $a_{11} = 0$ (*i.e.* $K^+$-p scattering in $S$ wave) and (iii) neglecting the possible distinction between $P_{\frac{3}{2}}$ and $P_{\frac{1}{2}}$ states in the $K^+$-n scattering.

These assumptions lead for the three reactions:

(I) $$K^+ + p = K^+ + p$$

(II) $$K^+ + n = K^+ + n$$

(III) $$K^+ + n = K^0 + p$$

to the following relationships for the angular, and total cross-section:

$$\frac{d\sigma}{d\Omega} = \lambda\!\!\!^{-2}(\alpha + \beta \cos\theta + \gamma \cos^2\theta), \qquad \sigma = 4\pi\lambda\!\!\!^{-2}(\alpha + \tfrac{1}{3}\gamma).$$

In these formula $\lambda$ is the de Broglie wave length of the $K$ and $\alpha$, $\beta$, $\gamma$ are respectively for the three reactions:

| I | II | III |
|---|---|---|
| $\alpha = a_{10}^2$ | $\alpha = \frac{1}{4}(a_{10} + a_{00})^2$ | $\alpha = \frac{1}{4}(a_{10} - a_{00})^2$ |
| $\beta = 0$ | $\beta = \frac{3}{2}(a_{10} + a_{00}) \cdot a_{01}$ | $\beta = -\frac{3}{2}(a_{10} - a_{00}) \cdot a_{01}$ |
| $\gamma = 0$ | $\gamma = \frac{9}{4}a_{01}^2$ | $\gamma = \frac{9}{4}a_{01}^2$ . |

Our analysis yielded the following figures:

$$a_{10} = -0.39 \pm 0.03 \qquad a_{00} = 0.0 \pm 0.007$$

$$a_{11} = 0 \text{ (hypothesis)} \qquad a_{01} = +0.16 \pm 0.05 .$$

The minus sign of $a_{10}$ means a repulsive nuclear potential $K^+$-p.
The ratio:

$$\frac{\sigma_{Kp}}{\sigma_{Kn} + \sigma_{CE}} = \frac{14.5 \pm 2.2}{11.4 \pm 2.8} = 1.27 \pm 0.37 ,$$

is clearly less than the value expected for a pure $T = 1$ state. The same conclusion is also given by the values for the total K-neutron cross-section reported by other authors (see table VII).

The presence of a $T = 0$ state can be concluded obviously from the $a_{Tl}$ evaluated in our analysis.

* * *

We are very grateful to Prof. W. F. Fry, who has generously given us the $K_{III}$ stack.

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Our thanks are due to Dr. Z. O'FRIEL for help in preparing the manuscript.

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## RIASSUNTO

Sono stati osservati 212.5 m di traccia di $K^+$ nell'intervallo energetico $40 \div 150$ MeV. Sono riportati i seguenti dati: 1) Collisioni $K^+$-idrogeno. 2) Sezione d'urto totale per collisioni anelastiche e per scambio carica. 3) Sezione d'urto angolare nel sistema del laboratorio tra 80 e 150 MeV. 4) Correlazione tra l'angolo di scattering e la perdita di energia, e tra quest'ultima e l'energia primaria. 5) Valore più attendibile del potenziale nucleare. Con questi dati è stata condotta un'analisi per ottenere la sezione d'urto totale $K^+$-nucleone, e la sezione d'urto differenziale nel centro di massa. In particolare, si è ricavata la sezione d'urto $K^+$-neutrone libero per scattering e quella per scambio carica. Sono state infine valutate le ampiezze di scattering relative ai due stati isotopici $T=1$ e $T=0$ nell'ipotesi che vi siano solo contributi delle onde $S$ e $P$. Lo schema logico della nostra analisi è riportato nella Tab. VI.

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# Sur la diagonalisation du modèle de Wentzel.

A. Chevalier et G. Rideau

*Institut H. Poincaré - Paris*

(ricevuto l'11 Giugno 1958)

**Résumé.** — Une méthode est utilisée qui donne des formes explicites pour la diagonalisation du modèle de Wentzel à une seule source. Les résultats sont discutés suivant les valeurs de la constante de couplage. Pour tout un domaine de valeurs est obtenu un comportement pathologique des opérateurs de Heisenberg, qui ne semble pas susceptible de réinterprétation physique simple. L'incidence sur la renormalisation de la constante de couplage est discutée.

A la suite des résultats obtenus par Pauli et Källén [1] dans l'étude du modèle de Lee, des travaux relativement récents ont été consacrés au modèle de Wentzel [2] ou à un modèle similaire [3], permettant de faire apparaître un comportement pathologique du modèle quand sont adoptées certaines prescriptions pour la renormalisation de la constante de couplage. Mais la situation ainsi mise en lumière nous semble différer sensiblement de celle qui prévaut dans le cas du modèle de Lee. Dans ce dernier, la nécessité d'introduire, par suite de la renormalisation, une constante de couplage imaginaire, conduisait d'une manière naturelle à l'emploi d'une métrique indéfinie et à des probabilités négatives [1]. Rien de semblable dans l'étude du modèle de Wentzel: l'hamiltonien reste dans tous les cas hermitique au sens habituel. Aussi a-t-il paru intéressant d'entreprendre une recherche précise et systématique de la diagonalisation de l'hamiltonien suivant les différentes valeurs de la constante de couplage. Les résultats ainsi obtenus nous amènent à penser

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[1] G. Källén et W. Pauli: *Det. Kgl. dansk. Vid. Selskab*, **30**, No. 7 (1955).

[2] *a*) W. Thirring: *Helv. Phys. Acta*, **28**, 344 (1955); *b*) E. Arnous: *Journ. Phys. et Rad.* **17**, 107 (1956).

[3] C. P. Enz: *Suppl. Nuovo Cimento*, **3**, 363 (1956).

que l'existence postulée d'opérateurs incidents et émergents équivaut à une limitation du domaine des valeurs admissibles de la constante de couplage, avec un cas limite très analogue à celui résultant de l'existence des fantômes dipolaires dans le modèle de Lee [4]. Tout le problème est, évidemment de savoir si ces résultats sont généraux ou, au contraire, sont liés aux caraactéristiques du modèle de Wentzel et cette question peut rester ouverte tant que des méthodes globales et mathématiquement sures n'auront pas été développées pour des modèles plus physiques tels que l'électrodynamique.

## 1. – Diagonalisation de l'hamiltonien de Wentzel.

Nous partons de l'hamiltonien suivant:

$$1) \quad \mathcal{H} = \int \mathrm{d}\boldsymbol{k}\,\omega(k)a^*(\boldsymbol{k})a(\boldsymbol{k}) + \lambda \int \frac{\mathrm{d}\boldsymbol{k}_1}{\sqrt{\omega(k_1)}} \frac{\mathrm{d}\boldsymbol{k}_2}{\sqrt{\omega(k_2)}} [u(\boldsymbol{k}_1)u(\boldsymbol{k}_2)a(\boldsymbol{k}_1)a(\boldsymbol{k}_2) + 2u^*(\boldsymbol{k}_1)u(\boldsymbol{k}_2)a^*(\boldsymbol{k}_1)a(\boldsymbol{k}_2) + u^*(\boldsymbol{k}_1)u^*(\boldsymbol{k}_2)a^*(\boldsymbol{k}_1)a^*(\boldsymbol{k}_2)],$$

où $u(\boldsymbol{k})$ est la transformée de Fourier de la fonction de source réelle $u(\boldsymbol{x})$, représentant un nucléon fixé à l'origine, $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$ étant les habituels opérateurs d'annihilation et de création des mésons de masse $m$, vérifiant:

$$(2) \qquad [a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}'), \qquad \omega(k) = \sqrt{\boldsymbol{k}^2 + m^2}.$$

Dans la suite, pour simplifier, nous poserons que $u(\boldsymbol{k})$ est seulement fonction de $|\boldsymbol{k}| = k$.

Suivant WENTZEL [5], nous cherchons à écrire $\mathcal{H}$ sous la forme:

$$(3) \qquad \mathcal{H} = \int \mathrm{d}k\omega(k)A^*(\boldsymbol{k})A(\boldsymbol{k}) + \mathrm{Cte},$$

où les $A(\boldsymbol{k})$, $A^*(\boldsymbol{k})$ sont des combinaisons linéaires des $a(k)$ et $a^*(k)$:

$$(4) \qquad \begin{cases} A(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}'\Gamma_1(\boldsymbol{k}, \boldsymbol{k}')a(\boldsymbol{k}') + \int \mathrm{d}\boldsymbol{k}'\Gamma_2(\boldsymbol{k}, \boldsymbol{k}')a^*(\boldsymbol{k}') \\ A^*(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}'\Gamma_1^*(\boldsymbol{k}, \boldsymbol{k}')a^*(\boldsymbol{k}') + \int \mathrm{d}\boldsymbol{k}'\Gamma_2^*(\boldsymbol{k}, \boldsymbol{k}')a(\boldsymbol{k}'), \end{cases}$$

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[4] W. HEISENBERG: *Nucl. Phys.*, **4**, 532 (1957).

[5] G. WENTZEL: *Helv. Phys. Acta*, **15**, 111 (1942); A. KLEIN et B. H. MCCORMICK: *Phys. Rev.*, **98**, 1428 (1955).

avec:

$$[A(\boldsymbol{k}), A^*(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}') \,, \tag{5}$$

tous autres commutateurs étant nuls.

Partant de:

$$[\mathcal{H}, A(\boldsymbol{k})] = -\omega(k) A(\boldsymbol{k}) \tag{6}$$

et identifiant dans les deux membres les coefficients des $a(\varkappa)$ et $a^*(\varkappa)$ nous obtenons le système:

$$\left\{\begin{aligned} &\Gamma_1(\boldsymbol{k}, \boldsymbol{l})\omega(l) + 2\lambda \frac{u(l)}{\sqrt{\omega(l)}} \cdot \\ &\qquad \cdot \int \mathrm{d}\boldsymbol{k}' \left[ \Gamma_1(\boldsymbol{k}, \boldsymbol{k}') \frac{u^*(k')}{\sqrt{\omega(k')}} - \Gamma_2(\boldsymbol{k}, \boldsymbol{k}') \frac{u(k')}{\sqrt{\omega(k')}} \right] = \omega(k) \Gamma_1(\boldsymbol{k}\boldsymbol{l}) \\ &\Gamma_2(\boldsymbol{k}, \boldsymbol{l})\omega(l) - 2\lambda \frac{u^*(l)}{\sqrt{\omega(l)}} \cdot \\ &\qquad \cdot \int \mathrm{d}\boldsymbol{k}' \left[ \Gamma_1(\boldsymbol{k}, \boldsymbol{k}') \frac{u^*(k')}{\sqrt{\omega(k')}} - \Gamma_2(\boldsymbol{k}, \boldsymbol{k}') \frac{u(k')}{\sqrt{\omega(k')}} \right] = -\omega(k) \Gamma_2(\boldsymbol{k}, \boldsymbol{l}) \,, \end{aligned}\right. \tag{7}$$

dont la résolution se fait en exprimant $\Gamma_1(\boldsymbol{k}\boldsymbol{l})$, $\Gamma_2(\boldsymbol{k}\boldsymbol{l})$ en fonction de

$$\int \mathrm{d}k' \left[ \Gamma_1(\boldsymbol{k}, \boldsymbol{k}') \frac{u^*(k')}{\sqrt{\omega(k')}} - \Gamma_2(\boldsymbol{k}, \boldsymbol{k}') \frac{u(k')}{\sqrt{\omega(k')}} \right] ,$$

cette dernière quantité étant alors déterminée immediatement par l'équation linéaire résultant de sa définition. Le résultat final est:

$$\Gamma_1(\boldsymbol{k}, \boldsymbol{l}) = \delta(\boldsymbol{k} - \boldsymbol{l}) + \\ + 2\lambda i \cdot \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{1}{1 - 4\lambda\mu(\omega(k))} \int_{-\infty}^{0} \mathrm{d}u \exp\left[iu(\omega(k) - \omega(l))\right] \frac{u(l)}{\sqrt{\omega(l)}} \,, \tag{9}$$

$$\Gamma_2(\boldsymbol{k}, \boldsymbol{l}) = 2\lambda i \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{1}{1 - 4\lambda\mu(\omega(k))} \int_{\infty}^{0} \mathrm{d}u \exp\left[iu(\omega(k) + \omega(l))\right] \frac{u^*(l)}{\sqrt{\omega(l)}} \,, \tag{10}$$

avec

$$\mu(\omega(k)) = \int_{-\infty}^{0} \mathrm{d}u \int \mathrm{d}\boldsymbol{l} \frac{|u(l)|^2}{\omega(l)} \sin u\omega(l) \exp\left[iu\omega(k)\right] ,$$

## 2. – Validité de (9) et (10).

La méthode précédente ne peut être considérée comme une déduction rigoureuse, car l'introduction du terme $\int_{-\infty}^{0} du \exp[iu(\omega(k)-\omega(l))]$ représente un certain choix dans l'ensemble des définitions possibles de la division par $[\omega(k)-\omega(l)]$ effectuée dans la première équation (7). De plus nous ne savons pas si les relations de commutation (6) sont satisfaites. En fait, leur vérification ne pose pas de problèmes particuliers et nous renvoyons à (A.I) pour la démonstration.

Il n'en va pas de même pour la vérification de (3) et, sans entrer dans les détails des calculs, pour lesquels nous renvoyons à (A.II), nous allons indiquer la marche de la vérification.

*a*) Nous calculerons pour commencer les termes en $a(\boldsymbol{l})a(\boldsymbol{l}')$ tels qu'ils apparaissent dans le second membre de (3). Ces termes ont pour coefficient sous le signe d'intégration

$$\frac{1}{2}\int \mathrm{d}\boldsymbol{k}\,\omega(k)[\Gamma_2^*(\boldsymbol{kl})\Gamma_1(\boldsymbol{kl}')+\Gamma_2^*(\boldsymbol{kl}')\Gamma_1(\boldsymbol{k},\boldsymbol{l})], \tag{11}$$

où le facteur $\frac{1}{2}$ tient compte de la symétrie en $\boldsymbol{l}$, $\boldsymbol{l}'$ du produit $a(\boldsymbol{l})a(\boldsymbol{l}')$. Après des manipulations simples permettant de mettre en facteur la quantité $\int_{-\infty}^{0} du \exp[-iu(\omega(l)+\omega(l'))]$, (11) s'écrit

$$\lambda\frac{u(l)}{\sqrt{\omega(l)}}\frac{u(l')}{\sqrt{\omega(l')}}\int_{-\infty}^{0} du \exp[-iu(\omega(l)+\omega(l'))]\cdot \tag{12}$$

$$\cdot\left\{-\frac{i\omega(l')}{1-4\lambda\mu^*(\omega(l'))}+4\lambda\int \mathrm{d}\boldsymbol{k}\int_{-\infty}^{0} du\,\frac{|u(k)|^2}{|1-4\lambda\mu(\omega(k))|^2}\cos u\omega(k)\exp[-iu\omega(l')]-\right.$$

$$\left.-\frac{i\omega(l)}{1-4\lambda\mu^*(\omega(l))}+4\lambda\int \mathrm{d}\boldsymbol{k}\int_{-\infty}^{0} du\,\frac{|u(k)|^2}{|1-4\lambda\mu(\omega(k))|^2}\cos u\omega(k)\exp[-iu\omega(l)]\right\},$$

ce qui ramène le calcul à l'évaluation de

$$I=4\lambda\int \mathrm{d}\boldsymbol{k}\int_{-\infty}^{0} du\,\frac{|u(k)|^2}{|1-4\lambda\mu(\omega(k))|^2}\cos u\omega(k)\exp[-iu\omega(l)]\,. \tag{13}$$

On utilise:

$$\frac{4\lambda}{|1-4\lambda\mu(\omega(k))|^2}=\frac{1}{\mu(\omega(k))-\mu^*(\omega(k))}\left\{\frac{1}{1-4\lambda\mu(\omega(k))}-\frac{1}{1-4\lambda\mu^*(\omega(k))}\right\},$$

$$\mu(\omega(k))-\mu^*(\omega(k))=4i\pi^2 k\,|u(k)|^2$$

et on remarque que:

$$(14)\qquad \begin{cases} 1 - 4\lambda\mu(\omega(k)) = F^{-}(\omega(k)) , \\ 1 - 4\lambda\mu^{*}(\omega(k)) = F^{+}(\omega(k)) , \end{cases}$$

la fonction de variable complexe $F(z)$ étant donnée par

$$(15)\qquad F(z) = 1 + 4\lambda \int \mathrm{d}\boldsymbol{l}\, \frac{|u(l)|^2}{\omega(l)^2 - z^2} .$$

C'est une fonction analytique dans tout le plan complexe à l'exception des coupures $m < z < +\infty$ et $-\infty < z < -m$ avec les définitions habituelles:

$$F^{\pm}(z) = \lim_{\varepsilon \to 0} F(z \pm i\varepsilon) .$$

L'introduction de $F(z)$ permet de calculer $I$ par intégration dans le plan complexe, le résultat obtenu dépendant de l'existence ou de la non existence de zéros de $F(z)$. L'étude de ces zéros se résume dans le tableau ci-dessous:

Tableau I.

| $\lambda$ | $-\infty$ région 3 $-\dfrac{1}{4\int \mathrm{d}\boldsymbol{l}\, \dfrac{(u(l))^2}{\omega(l)^2}}$ | région 2 $-\dfrac{1}{4\int \mathrm{d}\boldsymbol{l}\, \dfrac{(u(l))^2}{l^2}}$ | région $1 + \infty$ |
|---|---|---|---|
| $F(z)$ | deux zéros $\pm i\varrho$ | racine double $z = 0$ | deux zéros $\pm\sigma$ $\sigma < m$ | pas de zéros |

Les valeurs correspondantes de (13) sont données dans (A.II); en les reportant dans (12), on déduit les valeurs suivantes de(11):

Tableau II.

| | |
|---|---|
| région 1 | $\lambda \dfrac{u(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}}$ |
| région 2 | $\lambda \dfrac{u(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} - \dfrac{u(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} \dfrac{\sigma^2 - \omega(l)\,\omega(l')}{(\sigma^2 - \omega(l)^2)(\sigma^2 - \omega(l')^2)} \dfrac{1}{4\int \mathrm{d}\boldsymbol{k}\, \dfrac{(u(k))^2}{(\omega(k^2 - \sigma^2)^2}}$ |
| région 3 | $\lambda \dfrac{u(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} + \dfrac{u(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} \dfrac{\varrho^2 + \omega(l)\,\omega(l')}{(\varrho^2 + \omega(l)^2)(\varrho^2 + \omega(l')^2)} \dfrac{1}{4\int \mathrm{d}\boldsymbol{k}\, \dfrac{(u(k))^2}{(\varrho^2 + \omega(k)^2)^2}}$ |

Par suite de l'hermiticité du second membre de (3), les coefficients de $a^*(\boldsymbol{l})a^*(\boldsymbol{l}')$ se déduisent des précédents en prenant les imaginaires conjugués.

*b*) Le calcul des coefficients des termes en $a^*(\boldsymbol{l})a(\boldsymbol{l}')$ se conduit suivant les mêmes lignes que ci-dessus et se ramène de nouveau à l'évaluation de l'intégrale (13), avec les résultats suivants:

Tableau III.

| $\lambda$ | Coefficients de $a^*(\boldsymbol{l})\,a(\boldsymbol{l}')$ |
|---|---|
| région 1 | $2\lambda \dfrac{u^*(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}}$ |
| région 2 | $2\lambda \dfrac{u^*(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} - 2 \dfrac{u^*(l)}{\sqrt{\omega(l')}} \dfrac{u(l')}{\sqrt{\omega(l')}} \dfrac{\sigma^2 + \omega(l)\,\omega(l')}{(\sigma^2 - \omega(l)^2)(\sigma^2 - \omega(l')^2)} \dfrac{1}{4\int \mathrm{d}\boldsymbol{k} \dfrac{\lvert u(k)\rvert^2}{(\omega(k)^2 - \sigma^2)^2}}$ |
| région 3 | $2\lambda \dfrac{u^*(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} + 2 \dfrac{u^*(l)}{\sqrt{\omega(l)}} \dfrac{u(l')}{\sqrt{\omega(l')}} \dfrac{\varrho^2 - \omega(l)\,\omega(l')}{(\varrho^2 + \omega(l)^2(\varrho^2 + \omega(l')^2)} \dfrac{1}{4\int \mathrm{d}\boldsymbol{k} \dfrac{\lvert u(k)\rvert^2}{(\varrho^2 + \omega(k)^2)^2}}$ |

On doit remarquer qu'en ramenant à la forme $a^*(\boldsymbol{l})\,a(\boldsymbol{l}')$ les termes en $a(\boldsymbol{l}')\,a^*(\boldsymbol{l})$ du second membre de (3) nous faisons apparaître la constante réelle $\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{l}\,\omega(k)\,|\Gamma_2(\boldsymbol{k}\boldsymbol{l})|^2$.

En conclusion la forme pour $\mathcal{H}$ postulée dans (3) n'est valable que pour des valeurs de $\lambda$ situées dans la région 1. Dans ces cas on définira le vide physique $\Psi_0$ par:

$$A(\boldsymbol{k})\Psi_0 = 0\,.$$

Les vecteurs construits en faisant agir sur $\Psi_0$ des $A^*$ d'arguments quelconques et en nombre fini arbitraire sont des vecteurs propres de $\mathcal{H}$, la quantité $|\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{l}\,\omega(k)\,|\Gamma_2(\boldsymbol{k}\boldsymbol{l})|^2$ représentant alors la renormalisation de masse du vide.

## 3. – Diagonalisation dans les regions 2 et 3.

3·1. Région 2. – En posant:

$$N^2 = 4\sigma \int \mathrm{d}\boldsymbol{l}\, \frac{|u(l)|^2}{(\omega(l)^2 - \sigma^2)^2}\,,$$

introduisons les opérateurs $A(\sigma)$ et $A^*(\sigma)$ par:

$$(17)\quad \begin{cases} A(\sigma) = \dfrac{1}{N}\left\{\displaystyle\int \mathrm{d}\boldsymbol{k}' \frac{1}{\sigma - \omega(k')} \frac{u(k')}{\sqrt{\omega(k')}}\, a(\boldsymbol{k}') + \int \mathrm{d}\boldsymbol{k}' \frac{1}{\sigma + \omega(k')} \frac{u^*(k')}{\sqrt{\omega(k')}}\, a^*(\boldsymbol{k}')\right\}, \\ A^*(\sigma) = \dfrac{1}{N}\left\{\displaystyle\int \mathrm{d}\boldsymbol{k}' \frac{1}{\sigma - \omega(k')} \frac{u^*(k')}{\sqrt{\omega(k')}}\, a^*(\boldsymbol{k}') + \int \mathrm{d}\boldsymbol{k}' \frac{1}{\sigma + \omega(k')} \frac{u(k')}{\sqrt{\omega(k')}}\, a(\boldsymbol{k}')\right\}. \end{cases}$$

On vérifie alors immédiatement qu'à des constantes réelles près:

$$\mathcal{H} = \int \mathrm{d}\boldsymbol{k}\omega(k) A^*(\boldsymbol{k}) A(\boldsymbol{k}) + \sigma A^*(\sigma) A(\sigma)\,,$$

avec les relations de commutation:

$$(18)\quad \begin{cases} [A(\sigma), A^*(\sigma)] = 1\,, \\ [A(\sigma), A(\boldsymbol{k})] = [A(\sigma), A^*(\boldsymbol{k})] = 0\,, \\ [A^*(\sigma), A(\boldsymbol{k})] = [A^*(\sigma), A^*(\boldsymbol{k})] = 0\,. \end{cases}$$

$A(\sigma)$, $A^*(\sigma)$ peuvent donc respectivement être considérés comme les annihilateur et créateur d'un état d'énergie $\sigma$ qui, par suite de $\sigma < m$, est un état lié du système ([2b]); le vide physique $\Psi_0$ est défini par:

$$A(\boldsymbol{k})\Psi_0 = A(\sigma)\Psi_0 = 0$$

et on construit les differents vecteurs propres de $\mathcal{H}$ en faisant agir sur $\Psi_0$ des $A^*(\boldsymbol{k})$ et $A^*(\sigma)$ en nombre quelconque.

3·2. Région 3. – Posons alors:

$$(19)\quad A(\pm\varrho) = \int \mathrm{d}\boldsymbol{k}' \frac{1}{\pm\varrho + i\omega(k')} \frac{u(k')}{\sqrt{\omega(k')}}\, a(\boldsymbol{k}') + \int \mathrm{d}\boldsymbol{k}' \frac{1}{\pm\varrho + i\omega(k')} \frac{u^*(k')}{\sqrt{\omega(k')}}\, a^*(\boldsymbol{k}').$$

On vérifie que

$$(20)\quad \mathcal{H} = \int \mathrm{d}\boldsymbol{k}\omega(k) A^*(\boldsymbol{k}) A(\boldsymbol{k}) + \frac{1}{n^2} A(-\varrho) A(\varrho) - i\frac{\varrho}{2} + \text{Cte réelle}\,,$$

$$n^2 = 4\int \mathrm{d}\boldsymbol{l} \frac{|u(l)|^2}{(\varrho^2 + \omega(l)^2)^2}\,.$$

$A(\pm\varrho)$ est self-adjoint avec la relation de commutation:

$$(21)\quad [A(-\varrho), A(\varrho)] = i\varrho n^2\,,$$

les commutateurs de $A(\pm\varrho)$ avec les $A(\boldsymbol{k})$ et $A^*(\boldsymbol{k})$ étant nuls.

D'après un théorème général, l'espace de Hilbert où agissent $A(\varrho)$ et $A(-\varrho)$ est isomorphe à un espace de fonctions de carré sommable $f(x)$, $A(-\varrho)/n\sqrt{\varrho}$ étant alors isomorphe à la multiplication par $x$, $A(\varrho)/n\sqrt{\varrho}$ à l'opé-

reteur $-i\,(\mathrm{d}/\mathrm{d}x)$. Cet isomorphisme permet de déterminer les valeurs propres de l'opérateur hermitique $[(1/n^2)A(-\varrho)A(l)-i(\varrho/2)]$ et l'on constate alors qu'elles forment un spectre continu étendu de $-\infty$ à $+\infty$. Cette situation déjà rencontrée par ENZ ([3]) dans l'étude du couplage par paires de photons entraine l'absence d'un vecteur du vide et le caractère non défini positif de l'hamiltonien. Nous verrons de plus qu'il n'est alors plus possible d'introduire des opérateurs incidents et émergents, ainsi que l'avait déjà signalé ARNOUS ([2b]), d'où l'absence de matrice $S$.

3·3. Cas limite. – C'ezt le cas où $F(z)$ est nulle pour $z=0$. A des constantes réelles près $\mathcal{H}$ prend la forme:

$$\mathcal{H}=\int \mathrm{d}\boldsymbol{k}\,\omega(k)A^*(\boldsymbol{k})A(\boldsymbol{k})+A^2, \tag{22}$$

où

$$A=\frac{i}{\nu}\int\frac{\mathrm{d}\boldsymbol{k}'}{\omega(k')^{\frac{3}{2}}}\{u(k')a(\boldsymbol{k}')-u^*(k')a^*(\boldsymbol{k}')\}, \tag{23}$$

$$\nu^2=4\int \mathrm{d}\boldsymbol{k}\,\frac{|u(k)|^2}{\omega(k)^4}. \tag{24}$$

Introduisant l'opérateur $B$ par

$$B=\frac{1}{\nu}\int\frac{\mathrm{d}k'}{\omega(k')^{\frac{3}{2}}}\{u(k')a(\boldsymbol{k}')+u^*(k')a^*(\boldsymbol{k})'\}, \tag{25}$$

nous aurons la relation de commutation

$$[A,B]=\frac{i}{2}, \tag{26}$$

et en reprenant le raisonnement fait plus haut dans le cas $b$), nous verrons que $A$ est isomorphe, à des coefficients près, à la multiplication par $x$ dans un expace de fonctions de carré sommable. D'où de nouveau, un spectre continu de 0 à $+\infty$. L'hamiltonien reste défini positif, mais on ne peut définir le vide. L'étude des opérateurs de Heisenberg nous permettra de préciser les caractéristiques de ce cas limite.

## 4. – Les opérateurs de Heisenberg.

Pour calculer les opérateurs de Heisenberg

$$\begin{cases} a(\boldsymbol{kt}) &= \exp[it\mathcal{H}]\,a(\boldsymbol{k})\exp[-it\mathcal{H}], \\ a^*(\boldsymbol{kt}) &= \exp[it\mathcal{H}]\,a^*(\boldsymbol{k})\exp[-it\mathcal{H}], \end{cases} \tag{27}$$

il sera commode d'exprimer les $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$ en fonction des $A$, $A^*$ dans les différents cas. Les résultats sont

*Région* 1

$$(28)\quad \begin{cases} a(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' \Gamma_1^*(\boldsymbol{k}', \boldsymbol{k}) A(\boldsymbol{k}') - \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_2(\boldsymbol{k}', \boldsymbol{k}), \\ a^*(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_1(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_2^*(\boldsymbol{k}', \boldsymbol{k}), \end{cases}$$

*Région* 2

$$(29)\quad \begin{cases} a(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' \Gamma_1^*(\boldsymbol{k}', \boldsymbol{k}) A(\boldsymbol{k}') - \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_2(\boldsymbol{k}', \boldsymbol{k}) + \\ \qquad + \frac{1}{N} \frac{1}{\sigma - \omega(k)} \frac{u^*(k)}{\sqrt{\omega(k)}} A(\sigma) - \frac{1}{N} \frac{1}{\sigma + \omega(k)} \frac{u^*(k)}{\sqrt{\omega(k)}} A^*(\sigma), \\ a^*(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_1(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_2^*(\boldsymbol{k}', \boldsymbol{k}) + \\ \qquad + \frac{1}{N} \frac{1}{\sigma - \omega(k)} \frac{u(k)}{\sqrt{\omega(k)}} A^*(\sigma) - \frac{1}{N} \frac{1}{\sigma + \omega(k)} \frac{u(k)}{\sqrt{\omega(k)}} A(\sigma). \end{cases}$$

*Région* 3

$$(30)\quad \begin{cases} a^*(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_1(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_2^*(\boldsymbol{k}', \boldsymbol{k}) - \\ \qquad - \frac{1}{i\varrho n^2} \frac{1}{\varrho + i\omega(k)} \frac{u(k)}{\sqrt{\omega(k)}} A(\varrho) - \frac{1}{i\varrho n^2} \frac{1}{\varrho - i\omega(k)} \frac{u(k)}{\sqrt{\omega(k)}} A(-\varrho), \\ a(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_1^*(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_2(\boldsymbol{k}', \boldsymbol{k}) + \\ \qquad + \frac{1}{i\varrho n^2} \frac{1}{\varrho + i\omega(k)} \frac{u^*(k)}{\sqrt{\omega(k)}} A(-\varrho) + \frac{1}{i\varrho n^2} \frac{1}{\varrho - i\omega(k)} \frac{u^*(k)}{\sqrt{\omega(k)}} A(\varrho), \end{cases}$$

*Cas limite*

$$a(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_1^*(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_2(\boldsymbol{k}', \boldsymbol{k}) + \frac{2}{iv} \frac{u^*(k)}{\omega(k)^{\frac{1}{2}}} A + \frac{2}{v} \frac{u^*(k)}{\omega(k)^{\frac{1}{2}}} B,$$

$$a^*(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}' A^*(\boldsymbol{k}') \Gamma_1(\boldsymbol{k}', \boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}' A(\boldsymbol{k}') \Gamma_2^*(\boldsymbol{k}', \boldsymbol{k}) - \frac{2}{iv} \frac{u(k)}{\omega(k)^{\frac{1}{2}}} A + \frac{2}{v} \frac{u(k)}{\omega(k)^{\frac{1}{2}}} B.$$

Ces formulent s'établissent aisèment en formant les commutateurs de $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$ avec $A^*(\boldsymbol{l})$, $A(\boldsymbol{l})$, $A(\sigma)$, $A^*(\sigma)$, $A(-\varrho)$, $A(\varrho)$, $A$, $B$ successivement.

L'obtention de $a(\boldsymbol{k}, t)$ $a^*(\boldsymbol{k}, t)$ suivant les différents cas est maintenant

immédiate en utilisant:

$$\begin{aligned}
\exp[it\mathcal{H}]A(\boldsymbol{k})\exp[-it\mathcal{H}] &= \exp[-it\omega(k)]A(\boldsymbol{k}),\\
\exp[it\mathcal{H}]A^*(\boldsymbol{k})\exp[-it\mathcal{H}] &= \exp[it\omega(k)A^*(\boldsymbol{k}),\\
\exp[it\mathcal{H}]A(\sigma)\exp[-it\mathcal{H}] &= \exp[-i\sigma t]A(\sigma),\\
\exp[it\mathcal{H}]A^*(\sigma)\exp[-it\mathcal{H}] &= \exp[it\sigma]A^*(\sigma),\\
\exp[it\mathcal{H}]A(\varrho)\exp[-it\mathcal{H}] &= \exp[-\varrho t]A(\varrho),\\
\exp[it\mathcal{H}]A(-\varrho)\exp[-it\mathcal{H}] &= \exp[\varrho t]A(-\varrho),\\
\exp[it\mathcal{H}]A\exp[-it\mathcal{H}] &= A,\\
\exp[i\mathcal{H}t]B\exp[-it\mathcal{H}] &= B - tA.
\end{aligned}$$

Pour la région 1 on trouve ainsi:

$$a(\boldsymbol{k},t) = \int \mathrm{d}\boldsymbol{k}'\exp[-it\omega(k)]A(\boldsymbol{k}')\Gamma_1^*(\boldsymbol{k}',\boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}'\exp[it\omega(k')]A^*(\boldsymbol{k}')\Gamma_2(\boldsymbol{k}',\boldsymbol{k}) =$$

$$= \exp[-it\omega(k)]\left[A(\boldsymbol{k}) - 2i\lambda\frac{u^*(k)}{\sqrt{\omega(k)}}\int \mathrm{d}\boldsymbol{k}'\int_{-\infty}^{t}\mathrm{d}u\exp[iu(\omega(k)-\omega(k'))]\cdot\frac{u(k')}{\sqrt{\omega(k)}}\cdot\right.$$

$$\left.\cdot\frac{A(\boldsymbol{k}')}{1-4\lambda\mu^*(\omega(k))} - 2i\lambda\frac{u^*(k)}{\sqrt{\omega(k)}}\int \mathrm{d}\boldsymbol{k}'\int_{-\infty}^{t}\mathrm{d}u\exp[+iu(\omega(k)+\omega(k'))]\cdot\right.$$

$$\left.\cdot\frac{u^*(k')}{\sqrt{\omega(k')}}\frac{A^*(\boldsymbol{k}')}{1-4\lambda u(\omega(k'))}\right].$$

Les opérateurs incidents et émergents sont définis comme:

$$a_{\substack{\mathrm{in}\\\mathrm{out}}}(\boldsymbol{k}) = \lim_{t\to\mp\infty}\exp[it\omega(k)]a(\boldsymbol{k},t),$$

soit

$$(32)\qquad \begin{cases} a_{\mathrm{in}}(\boldsymbol{k}) = A(\boldsymbol{k}),\\ a_{\mathrm{out}}(\boldsymbol{k}) = A(\boldsymbol{k}) - \dfrac{4i\pi\lambda u^*(k)}{\sqrt{\omega(k)}}\displaystyle\int \mathrm{d}\boldsymbol{k}'\delta(\omega(k)-\omega(k'))\frac{u(k')}{\sqrt{\omega(k')}}\frac{A(\boldsymbol{k}')}{1-4\lambda\mu^*(\omega(k'))}. \end{cases}$$

Le vide final est donc identique au vide initial et l'écriture des amplitudes de transition est immédiate. Ainsi l'amplitude de transition d'un méson de moment initial $\boldsymbol{k}_i$ vers un méson de moment final $\boldsymbol{k}_f$ est donnée par la formule (26):

$$(33)\qquad \delta(\boldsymbol{k}_i-\boldsymbol{k}_f) + 4i\pi\lambda\frac{u(k_i)}{\sqrt{\omega(k_i)}}\delta(\omega(k_i)-\omega(k_f))\frac{u^*(k_f)}{\sqrt{\omega(k_i)}}\frac{1}{1-4\lambda u(\omega(k_f))}.$$

Nous reviendrons plus loin sur cette formule en abordant le problème des renormalisations.

Dans le cas de la région 2, les exponentielles $\exp[i(\omega(k) \pm \sigma)t]$ tendant vers zéro quando $t \to \pm\infty$, les formules (32) s'obtiennent encore, mais pour définir une base complète de l'espace de Hilbert il faut ajouter l'opérateur $A(\sigma)$ dont il est immédiat que les formes incidentes et émergentes sont identiques à lui-même, correspondant bien au fait que $A^*(\sigma)\Psi_0$ décrit un état lié. Les amplitudes de transition entre $p$ mésons initiaux et $q$ mésons finaux sont simplement multipliées par un $\delta_{nm}$ si initialement le système contient le vecteur $A^*(\sigma)^n\Psi_0$ et finalement le vecteur $A^*(\sigma)^m\Psi_0$.

La situation est totalement modifiée dans la région 3 par la présence dans $a(\boldsymbol{k}, t)$ des termes en $\exp[-\varrho t]$ et $\exp[\varrho t]$, qui conduisent l'un ou l'autre, à une croissance infinie pour $t \to -\infty$ ot $t \to +\infty$. Les contributions de ces termes ne pourraient être éliminées qu'en imposant la condition $A(\varrho)\Psi = 0$ pour les vecteurs initiaux et la condition $A(-\varrho)\Psi = 0$ pour les vecteurs finaux. Mais d'après l'isomorphisme indiqué plus haut $A(\varrho)$, $A(-\varrho)$ ont un spectre continu et il n'y a pas de vecteur normé vérifiant de telles conditions. D'où la conclusion annoncée dans l'introduction.

Reste le cas limite où apparaissent un terme indépendant du temps et un terme linéaire en $t$ (4). Le premier est éliminé par le passage à la limite; quant au second, il peut l'être au sens des distributions. Pour cela, au lieu de $a(\boldsymbol{k}, t)$, considérons l'opérateur $a_\varphi(\boldsymbol{k}t) = \int \mathrm{d}\boldsymbol{k}\varphi(\boldsymbol{k})\, a(\boldsymbol{k}, t)$ dont les formes incidentes et émergentes seront évidemment données par:

$$\lim_{t \to \mp\infty} \int \mathrm{d}\boldsymbol{k}\varphi(\boldsymbol{k})\, a(\boldsymbol{k}, t) \exp[it\omega(k)] \,.$$

Si, alors, $\varphi(\boldsymbol{k})$ est choisie telle que:

$$\lim_{t \to \pm\infty} t \int \mathrm{d}\boldsymbol{k}\varphi(\boldsymbol{k}) \frac{u^*(k)}{\omega(k)^{\frac{3}{2}}} \exp[it\omega(k)] = 0 \,,$$

nous aurons fait disparaître dans le passage à la limite les termes linéaires en $t$. Ce procédé revient à placer les mésons initiaux et finaux dans des trains d'onde convenablement choisis, c'est-à-dire à s'imposer des restrictions sur les vecteurs d'états physiquement accessibles. Reste la difficulté liée à la non existence du vide, dont nous ne savons pas si elle peut être levée par cette méthode.

Une autre manière d'aborder le problème et qui, précisément, permettrait de fixer le vecteur du vide, serait de soumettre les vecteurs initiaux et finaux à l'équation $A\Psi = 0$. Mais le spectre continu de $A$, là encore interdit d'y recourir.

En définitive il ressort de cette analyse que les seuls cas ayant un sens

physique déterminé correspondent aux valeurs de $\lambda$ situés dans les régions 1 et 2. Dans les autres cas, la construction des vecteurs de base initiaux et terminaux se heurtet à des ambiguités ou des difficultés qu'il ne semble pas possible de lever sans sortir largement du cadre de la théorie.

## 5. – Renormalisation de la constante de couplage.

Nous avons conservé tout au long des calculs la fonction de source $u(k)$ sur laquelle étaient faites implicitement des hypothèses assurant la convergence de toutes les intégrales écrites. Si nous voulons approcher de plus près les conditions habituelles de la théorie quantique des champs, nous aurons à considérer comme source un nucléon ponctuel, autrement dit à prendre partout $u(k) = 1$. Si l'on n'effectue corrélativement aucune modification de la constante de couplage, la quantité $\lambda/(1 - 4\lambda\mu(\omega(k)))$ tendra vers zéro. Il en résulterait, d'après (32), l'identité des opérateurs incidents et émergents et l'absence de toute diffusion. Il est donc nécessaire de renormaliser $\lambda$ de telle sorte que l'on obtienne pour la quantité en question une limite bien déterminée et non nulle.

Pour simplifier les choses, prenons $|u(k)|^2 = K^2/(K^2 + k^2)$; $K$ représentant en quelque sorte un cut-off des énergies. Quand $K \to \infty$, $|u(k)|^2 \to 1$ au sens des distributions. Pour ce choix

$$\frac{\lambda}{1 - 4\lambda\mu(\omega(k))} = \lambda \frac{K + ik}{K(1 + 8\lambda\pi^2 K) + ik} \cdot \tag{34}$$

En prenant pour $\lambda$ le développement

$$\lambda = \frac{a_1}{K} + \frac{a_2}{K^2} + \ldots$$

(34) tendra vers une limite non nulle pour $a_1 = -1/8\pi^2$, cette limite étant:

$$-\frac{1}{8\pi^2} \frac{1}{8\pi^2 a_2 + ik}, \tag{35}$$

la constante $a_2$ devant être fixée par l'expérience.

Si, avant le passage à la limite, nous étions dans la région 1, $\lambda$ était supérieur à $-(4\int d\boldsymbol{l}\,|u(l)|^2/l^2)^{-1} = -1/8\pi^2 K$. Alors, avec $a_2$ positif, nous serons continuellement dans cette région et, à la limite, nous décrirons un nucléon sans état lié.

Si, au contraire, avant le passage à la limite, nous étions dans la région 2, nous avions $-1/8\pi^2 K - m/8\pi^2 K^2 < \lambda < -1/8\pi^2 K$. Avec l'inégalité $-m/8\pi^2 < a_2 < 0$, nous resterons dans cette région et, à la limite, nous aurons un nucléon avec état lié, dont l'énergie $\sigma$ se déduit de $a_2$ par:

$$a_2 = -1/8\pi^2 \cdot \sqrt{m^2 - \sigma^2}\,.$$

Enfin, si $a_2 < -m/8\pi^2$, nous aurons à la limite un système pathologique, la quantité $\varrho$ se déduisant de $a_2$ par:

$$a_2 = -1/8\pi^2 \cdot \sqrt{m^2 + \varrho^2}\,.$$

Cette rapide analyse met bien en lumière l'influence de la valeur de la renormalisation de charge sur le caractère de la description finalement obtenue, ainsi que l'a déjà signalé ARNOUS [2b].

Ajoutons enfin que, dans le modèle étudié, la donnée expérimentale d'un état lié d'énergie $\sigma$, équivaut à la donnée de la renormalisation de la charge, puisque, par suite de l'équation:

$$1 + 4\lambda \int \mathrm{d}\boldsymbol{l}\, \frac{|u(l)|^2}{\omega(l)^2 - \sigma^2} = 0\,,$$

il vient: $\lambda = -1/8\pi^2 K - 1/8\pi^2 K^2 \sqrt{m^2 - \sigma^2}$.

La même remarque s'applique à la donnée d'un état pathologique. Autrement dit, la théorie édifiée dans les conditions des régions 2 et 3 est une théorie libre, dès le départ, de tous infinis.

* * *

Nous sommes heureux d'exprimer nos remerciements au Dr. ARNOUS, dont le travail a été le point de départ du présent article, au Dr. JOUVET, pour de nombreuses et fructueuses discussions et au Centre National de la Recherche Scientifique dont l'aide financière nous a, en particulier, permis de mener à bien cet ouvrage.

## APPENDICE MATHÉMATIQUE

### 1. – Verification des relations (5).

*Vérification des relations* (5).

A partir de (9) et (10), il vient:

$$[A(\boldsymbol{k}), A^*(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}') +$$

$$+ 2\lambda \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{u(k')}{\sqrt{\omega(k')}} \frac{1}{\omega(k) - \omega(k') - i\varepsilon} \left\{ \frac{1}{1 - 4\lambda\mu(\omega(k))} - \frac{1}{1 - 4\lambda\mu^*(\omega(k'))} \right\} +$$

$$+ 4\lambda^2 \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{u(k')}{\sqrt{\omega(k')}} \frac{1}{1 - 4\lambda\mu(\omega(k))} \frac{1}{1 - \lambda\mu^*(\omega(k'))} \int \mathrm{d}\boldsymbol{l}\, \frac{|u(l)|^2}{\omega(l)} \cdot$$

$$\cdot \left\{ \frac{1}{(\omega(k) - \omega(l) - i\varepsilon)(\omega(k') - \omega(k) + i\varepsilon)} - \frac{1}{(\omega(k) + \omega(l) - i\varepsilon)(\omega(k') + \omega(l) + i\varepsilon)} \right\}.$$

La dernière intégral écrite se met sous la forme

$$\overline{\omega(k)} - \overline{\omega(k')} - 2i\varepsilon \int \mathrm{d}\boldsymbol{l}\, |u(l)|^2 \left\{ \frac{1}{(\omega(k') + i\varepsilon)^2 - \omega(l)^2} - \frac{1}{(\omega(k) - i\varepsilon)^2 - \omega(l)^2} \right\}.$$

Par ailleurs il résulte de sa définition que:

$$\mu(\omega(k)) = -\int \mathrm{d}\boldsymbol{l}\, \frac{|u(l)|^2}{\omega(l)^2 - (\omega(k) - i\varepsilon)^2}.$$

D'où en définitive:

$$[A(\boldsymbol{k}), A^*(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}') +$$

$$+ 2\lambda \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{u(k')}{\sqrt{\omega(k')}} \frac{1}{\omega(k) - \omega(k') - i\varepsilon} \left\{ \frac{1}{1 - 4\lambda\mu(\omega(k))} - \frac{1}{1 - 4\lambda\mu^*(\omega(k'))} \right\} +$$

$$+ 8\lambda^2 \frac{u^*(k)}{\sqrt{\omega(k)}} \frac{u(k')}{\sqrt{\omega(k')}} \frac{1}{\omega(k) - \omega(k') - 2i\varepsilon} \frac{\mu^*(\omega(k')) - \mu(\omega(k))}{[1 - 4\lambda\mu(\omega(k))][1 - 4\lambda\mu^*(\omega(k'))]},$$

et il reste après réduction du second terme au meme dénominateur et passage à la limite $\varepsilon \to 0$:

$$[A(\boldsymbol{k}), A^*(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}'),$$

La vérification des deux autres relations de commutation se fait suivant les memes lignes.

## 2. – Calcul de l'intégrale (13).

Nous avons à calculer:

$$I = \lim_{\varepsilon \to 0} (\varepsilon - i\omega(l)) \int \mathrm{d}\boldsymbol{k} \frac{|u(k)|^2}{\mu(\omega(k)) - \mu^*(\omega(k))} \left\{ \frac{1}{F^-(\omega(k))} - \frac{1}{F^+(\omega(k))} \right\} \cdot$$

$$\cdot \frac{1}{(\varepsilon - i\omega(l))^2 + \omega(k)^2} =$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon - i\omega(l)}{i\pi} \int_0^\infty k\mathrm{d}k \cdot \left[ \frac{1}{F^-(\omega(k))} - \frac{1}{F^+(\omega(k))} \right] \frac{1}{(\varepsilon - i\omega(l)^2 + \omega(k)^2}$$

en tenant compte de $\mu(\omega(k)) - \mu^*(\omega(k)) = 4i\pi^2 k |u(k)|^2$.

Posant $x = \omega(k)$, nous aurons à calculer:

$$I = \lim_{\varepsilon \to 0} \frac{\varepsilon - i\omega(l)}{i\pi} \int_m^\infty x\, \mathrm{d}x \left[ \frac{1}{F^-(x)} - \frac{1}{F^+(x)} \right] \frac{1}{(\varepsilon - i\omega(l))^2 + x^2}.$$

Considérons l'intégrale $-(\varepsilon - i\omega(l))/(2i\pi)\int_G (z\mathrm{d}z)/F(z)\cdot(1/(\varepsilon - i\omega(l))^2 + z^2)$ avec le contour $G$ ci-contre. En tenant compte de $F^+(-z) = F(z)$, on vérifie que la contribution des coupures est égale à $I$, la contribution sur le pourtour du cercle donnant $-(\varepsilon - i\omega(l))$ puisque $F(z) \to I$ quand $|z| \to \infty$. Quatre cas sont à distinguer suivant la position des zéros de $F(z)$.

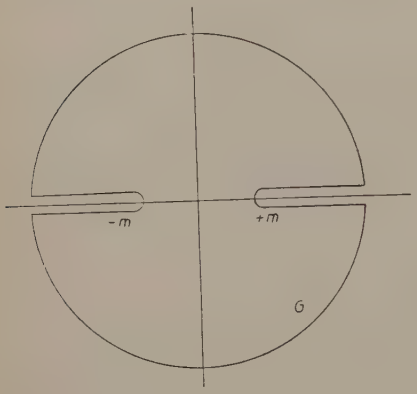


Fig. 1.

*a*) Région 1.

Les seuls pôles intérieurs au contour sont $z = \pm(\omega(l) + i\varepsilon)$. Après le passage à la limite $\varepsilon \to 0$:

$$I = -i\omega(l) + i\omega(l)/F^+(\omega(l)) .$$

*b*) Région 2.

Aux pôles du cas *a*) s'ajoutent les pôles $\pm\sigma$, avec, après passage à la limite:

$$I = -i\omega(l) + \frac{i\omega(l)}{F^+(\omega(l))} + \frac{i\omega(l)}{\lambda N^2}\,\frac{\sigma}{\sigma^2 - \omega(l)^2} .$$

*c*) Région 3.

Aux pôles du cas *a*) s'ajoutent les pôles $\pm i\varrho$ avec, après passage à la limite:

$$I = -i\omega(l) + \frac{i\omega(l)}{F^+(\omega(l))} - \frac{i\omega(l)}{\varrho^2 + \omega(l)^2}\,\frac{1}{\lambda n^2} .$$

*d*) Cas limite.

Aux pôles du cas *a*) s'ajoute un pôle double à l'origine, le résultat étant, après passage à la limite:

$$I = -i\omega(l) + \frac{i\omega(l)}{F^+(\omega(l))} + \frac{1}{i\omega(l)\lambda\nu^2} .$$

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RIASSUNTO (*)

Si utilizza un metodo che fornisce forme esplicite per la diagonalizzazione del modello di Wenzel a sorgente unica. Si discutono i risultati secondo i valori della costante di accoppiamento. Per un intero dominio di valori si ottiene un comportamento patologico degli operatori di Heisenberg che non sembra possibile di una semplice reinterpretazione fisica. Si discute l'incidenza della costante d'accoppiamento sulla rinormalizzazione.

(*) *Traduzione a cura della Redazione.*

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# A Note on the Apparent Inconsistency Among Measurements of Threshold Pion Interactions.

M. Cini, R. Gatto, E. L. Goldwasser (*) and M. Ruderman (+)

*Istituto di Fisica e Scuola di Perfezionamento dell'Università - Roma*
*Istituto Nazionale di Fisica Nucleare - Sezione di Roma*

(ricevuto il 16 Giugno 1958)

**Summary.** — The apparent inconsistency among threshold pion interactions is resolved by the use of extrapolation procedures for charge exchange and photoproduction amplitudes more in accord with theory than conventional linear extrapolations.

## 1. – Introduction.

With the improvement in accuracy of experiments involving pions at low energies and at zero energy it has become apparent that an inconsistency seems to exist among the results of these experiments when they are analyzed in conventional ways [1-3]. The pertinent measurements are:

(*) On sabbatical leave from the University of Illinois with grants from Guggenheim Foundation and Fulbright Program.

(+) John Simon Guggenheim Foundation Fellow on leave from the University of California, Berkeley.

(1) H. Bethe and F. de Hoffmann: *Mesons and Fields* (Evanston), Vol. II: *a*) p. 99; *b*) p. 174.

(2) J. M. Cassels: *Proceedings 7th Rochester Conference*, (1957), Sect. II, p. 1; J. M. Cassels, G. Fidecaro, A. M. Wetherell and J. R. Wormald: *Proc. Phys. Soc.*, A **70**, 405 (1957).

(3) A. M. Baldin: *Nuovo Cimento*, **8**, 569 (1958).

1) The Panofsky ratio:

$$R = \frac{w(\pi^- + p \to \pi^0 + n)}{w(\pi^- + p \to \gamma + n)} = 1.5 \pm 0.1 , \tag{1}$$

measured for zero kinetic energy negative pions ([4]);

2) The charge exchange transition rate

$$w(\pi^- + p \to \pi^0 + n) \tag{2}$$

measured at low energies and extrapolated to zero assuming a constant charge exchange amplitude ([5]);

3) The transition rate for photoproduction of positively charged pions from protons (and, from detailed balacing, its inverse)

$$w(\gamma + p \to \pi^+ + n) \tag{3}$$

which is also measured at low energies and must be extrapolated to threshold ([6,7]);

4) the ratio

$$r = \frac{w(\gamma + n \to \pi^- + p)}{w(\gamma + p \to \pi^+ + n)} , \tag{4}$$

which, in principle can be inferred from the observed $\pi^-/\pi^+$ ratio from photons on deuterons, extrapolated to a value of $1.87 \pm 0.13$ at threshold ([7]). According to BALDIN ([3]) this implies that at threshold $r \cong 1.4$ which is not in disagreement with the well founded theoretical argument which gives $r \cong 1.3$ ([8]).

However this value of $r$, together with the measurements (2) and (3) extrapolated to threshold leads to a calculated Panofsky ratio, $R$, of 2.5 in contrast to the measured value (1). This discrepancy has recently motivated the suggestion ([3]) that, perhaps, it is another neutral pion of isotopic spin zero ($\pi_0^0$)

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([1]) See ref. ([2]).

([5]) J. OREAR: *Nuovo Cimento*, **4**, 856 (1957).

([6]) G. BERNARDINI and E. L. GOLDWASSER: *Phys. Rev.*, **49**, 729 (1954); **95**, 857 (1954).

([7]) M. BENEVENTANO, G. BERNARDINI, D. CARLSON LEE, G. STOPPINI and L. TAU: *Nuovo Cimento*, **4**, 323 (1956).

([8]) G. F. CHEW, M. L. GOLDBERGER, F. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1345 (1057).

that is observed in the Panofsky experiment, and a crucial test has been proposed to test this hypothesis.

We wish to emphasize three points in connection with the above arguments which make the low energy data consistent without the introduction of a new particle and show that the introduction of such a particle is inconsistent with already existing experimental data.

We outline immediately below the three points which will be developed more fully in Sections **2**, 3 and 4:

*a*) The extrapolation to threshold of the transition rate (3) can be accomplished by taking into account the contribution to the *s*-wave of the well known direct photon-meson interaction term ([9])

$$2\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\varkappa}-\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{\epsilon}}{(\boldsymbol{\varkappa}-\boldsymbol{q})^2+\mu^2},$$

which vanishes at threshold but is not negligible at the energies where experiments are performed. This extrapolation is also consistent with more elaborate theoretical expressions given by dispersion theory, provided the magnitude of a quantitatavely unknown *s*-wave term believed to be qualitatively negligible, is assumed to be zero. The effect of the correction is to increase the threshold value of (3) given in ([7]) of about 30%. This also increases the coupling constant $f^2$ as calculated from photoproduction from 0.067 to 0.073.

*b*) The conventional extrapolation of the charge exchange amplitude is performed in accordance with the prescription that near threshold $(\alpha_1-\alpha_3)/k$ is independent of $\omega$ ($\omega=\sqrt{k^2+\mu^2}$). However at $\omega=0$ this amplitude can be proved to vanish so that a best fit to $(\alpha_1-\alpha_3)/k$ *vs.* $\omega$ should include this information. Indeed an *s*-wave effective range type approximation inferred from dispersion relations gives for the charge exchange amplitude

$$(5)\qquad (\alpha_1-\alpha_3)/k\sim\left[a\left(\frac{\omega}{\mu}\right)+b\left(\frac{\omega}{\mu}\right)^3\right]\frac{1}{1+(\omega/M)},$$

where $a$ and $b$ are constants, $a>0$, $a\gg b$. The fitting of low energy charge exchange data to the form (5) reduces the extrapolated threshold amplitude by 10%. An extrapolation linear in $\omega$ reduces it by 15%. At worst the form (5) gives some idea of the uncertainty inherent in the method of extrapolation. Actually it should lead to a more reliable value than previously found.

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([9]) G. F. Chew and F. Low: *Phys. Rev.*, **101**, 1579 (1956). The importance of this term in the analysis of photoproduction of charged pions has been stressed by M. J. Moravcsik: *Phys. Rev.*, **104**, 1451 (1956).

The combination of the changes described in *a*) and *b*), when applied to the cross-sections (2) and (3) reduce the calculated Panofsky ratio to a value consistent with experiment.

*c*) Present experimental data are already inconsistent with the suggested observation of a $\pi_0^0$ in the Panofsky experiment. This is proved by noticing that at higher energies the charge exchange *s*-wave scattering of $\pi^-$ by *p* will produce both $\pi_0^0$ and the conventional $\pi^0$. The partial cross section for $\pi_0^0$ is determined by the threshold values (1), (3) and (4). The partial cross-section for $\pi^0$ may be inferred from $\sigma(\pi^- + \mathrm{p} \to \pi^- + \mathrm{p})$, $\sigma(\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p})$ and the charge independence hypothesis. The total exchange scattering of $\pi^-$ into $\pi^0$ and $\pi_0^0$ at 20, 30 and 42 MeV predicted in this way is greatly in excess of that which is observed at these energies.

## 2. – Treatment of the photoproduction data.

The Panofsky branching ratio relates the transition rates for the processes

$$\pi^- + \mathrm{p} \to \pi^0 + \mathrm{n} \tag{1a}$$

$$\pi^- + \mathrm{p} \to \gamma + \mathrm{n} \tag{1b}$$

at zero kinetic energy for the $\pi^-$-meson.

In this part of the discussion we shall be concerned with the calculation of the transition rate for the process (1*b*) using the experimental information available from the process of photoproduction from protons (3). In Sect. 3 we shall calculate the same quantity for the process (1*a*). We shall then be in a position to calculate a value for the Panofsky ratio which can be compared to the measured value of $1.5 \pm 0.1$.

Since the $\pi^-$ absorption occurs at rest we shall be concerned only with $l = 0$ angular momentum states. There are, then, four problems involved in handling the photoproduction data:

1) The *s*-wave part of the measured cross-section $\sigma$ must be calculated.

2) The value of the $\pi^-/\pi^+$ ratio $r(\omega)$, for photoproduction from free neutrons and protons must be used to calculate the rate of transition for *s*-wave $\pi^-$ photoproduction from neutrons.

3) A standard detailed balancing calculation must be applied, in order to obtain the transition rate for the inverse process, radiative capture of $\pi^-$-mesons.

4) The radiative capture transition rate must be properly extrapolated to zero energy.

Before proceeding further in the study of these problems it will be useful to define the important terms which contribute to the low energy ($< 200$ MeV) photomeson production amplitude. The theoretical expression for differential cross-section can be assumed as follows ([9 bis]):

$$(6)\qquad \frac{d\sigma^{\pm}}{d\Omega} = \frac{2e^2f^2}{\mu^2}\cdot\frac{q\omega}{(1+(\omega/M))^2}\sum\left|\frac{1}{\sqrt{\varkappa\omega}}\left[(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})(1\mp\tfrac{1}{2}R(\omega)) + \right.\right.$$

$$\left.\left. + \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\varkappa}-\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{\epsilon}}{(\boldsymbol{\varkappa}-\boldsymbol{q})^2+1} + P(\omega)\right]\right|^2 = \frac{2e^2f^2}{\mu^2}W(\omega)\sum|\{\ldots\}|^2,$$

where the sum implies the average over nucleon spin and proton polarization and

$f^2$ = pion nucleon renormalized coupling constant,

$\mu$ = meson rest mass,

$q$ = meson momentum in center of mass system in units of $\mu$,

$\omega$ = meson energy in center of mass system in units of $\mu$,

$\varkappa$ = photon energy in center of mass system in units of $\mu$,

$M$ = nucleon rest mass in units of $\mu$,

$\hbar = c = 1$.

We shall refer to the terms inside the square bracket as the photoproduction amplitude. Here $\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}$ is commonly called the gauge invariance term $\pm\frac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})R(\omega)$ is the recoil term which gives rise to a different photoproduction cross-section for negative and positive meson from neutrons and protons respectively: its value is given by theory as $\pm[(g_p+g_n)/M]\omega(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})$, where $g_p$ and $g_n$ are the proton and neutron gyromagnetic ratios. The third term in the bracket is called the direct interaction term. $P(\omega)$ includes all the $p$-wave contributions. The square of the gauge invariance plus direct interaction terms in the amplitude, averaged over spins and polarization, contributes to the cross-section terms including

$$-\frac{v^2\sin^2\theta}{2\varkappa^2(1-v\cos\theta)^2},$$

($v$ = pion velocity in units of $c$). This term makes contributions to all partial waves.

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([9 bis]) Throughout this paper the magnitude of a quantitatively unknown $s$-wave term, suggested by the dispersion relation treatment, has been assumed to be zero. The term is hoped to be small, and within the framework of the presently available experimental data and previous analyses, it can offer nothing to relieve apparently existing inconsistencies.

The conventional analysis of the cross-section in terms of $s$- and $p$-waves is:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\pm} = A_0^{\pm}(\omega) \pm A_1^{\pm}(\omega)\cos\theta + A_2^{\pm}(\omega)\cos^2\theta\,, \tag{7}$$

where, neglecting $p$-wave recoil effects, the three coefficients may be written as follows

$$\begin{cases} A_0^{\pm}(\omega) = \dfrac{2e^2f^2}{\mu^2}\,\dfrac{W(\omega)}{\varkappa\omega}\left[a_{0S}^{\pm}(\omega) + a_{0P}(\omega)\right], \\ A_1^{\pm}(\omega) = -\dfrac{4e^2f^2}{\mu^2}\,\dfrac{W(\omega)}{\varkappa\omega}\,\mathrm{Re}\sqrt{a_{0S}^{\pm}(\omega)K(\omega)}\,, \\ A_2^{\pm}(\omega) = \dfrac{2e^2f^2}{\mu^2}\,\dfrac{W(\omega)}{\varkappa\omega}\left[K(\omega)^2 - a_{0P}(\omega)\right]. \end{cases} \tag{8}$$

Here
$a_{0S}^{\pm}$ = $s$-wave contribution,
$a_{0P}$ = $p$-wave, non spin flip, contribution,
$K(\omega)$ = $p$-wave, spin flip, contribution.

Since at low energies each of these quantities must be approximately real, knowing the three coefficients of the angular distribution, the three $s$- and $p$-wave components can be obtained. Furthermore a threshold theorem [12] can be applied to argue that

$$a_{0S}^{\pm}(\omega) \cong 1 \pm R(\omega)\,, \tag{9}$$

where $R(\omega)$ is of the order $\omega/M$.

After having separated the $s$-wave part of the cross-section at 170 MeV, BETHE and DE HOFFMANN [1] extrapolate it to threshold with the energy dependence predicted for the gauge invariance term alone.

This $s$- and $p$-wave analysis [10b] and the extrapolations that have been used before have thus explicitly neglected the particular angular distribution and energy dependence of the direct interaction term in the photoproduction amplitude (6). This term together with the gauge invariance and $s$-wave recoil terms have a « classical » origin. They do not depend upon dispersion relation calculations or upon any particular meson theory. The direct interaction term therefore rests on firm theoretical footing, and its existence has apparently

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[10] G. F. CHEW: *Phys. Rev.*, **95**, 1669 1(954).

[11] J. H. MALMBERG and C. S. ROBINSON: *Phys. Rev.*, **109**, 158 (1958); E. KNAPP, W. IMHOF, R. W. KENNEY and PEREZ MENDEZ: *Phys. Rev.*, **107**, 323 (1957).

[12] N. KROLL and M. RUDERMAN: *Phys. Rev.*, **93**, 233 (1954).

been verified experimentally (11). In addition to contributions to partial waves other than $s$ and $p$, this term has the property that it contributes to the $s$-wave amplitude a part which has an appreciable magnitude at 170 MeV, but goes to zero at threshold. To see the magnitude of this effect we can make a partial wave analysis of the gauge invariance plus direct interaction terms. We find for the $s$-wave amplitude ([8]):

$$F_s = 1 - \frac{1}{2}\left(1 + \frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right), \tag{10}$$

where the first term, 1, expresses the gauge invariance contribution alone. Evaluating this expression we find:

| $E_\gamma$ | $F_s$ | $I_s = F_s^2$ |
|---|---|---|
| threshold | 1 | 1 |
| 170 MeV | 0.94 | 0.89 |
| 200 MeV | 0.87 | 0.76 |

Thus, on this basis alone, a proper $s$-wave separation at 170 MeV would give an $s$-wave contribution proportional to $I_s = 0.89$ whereas the value $I_s = 1$ would be the correct one to use with an extrapolation to zero consistent with the energy dependence of the gauge invariance term. Actually the above argument cannot be taken as it stands, because it may be shown ([7]) that there is a large $s$-$d$ interference from these two terms of the amplitude, which changes radically the angular distribution analysis and the apparent $s$-wave contribution.

In view of the sound footing upon which the direct interaction term rests, it seems to be essential to take its effects into account quantitatively in separating out any $s$-wave and in making any extrapolation to zero. We therefore prefer to abandon the common method of analysis in $s$- and $p$-waves and instead to use a calculation based on the theoretical energy dependence of gauge invariance, recoil, and direct interaction terms. Theory predicts, and experiment is consistent with the fact that at 175 MeV the contribution of all other terms with the exception of the unknown $s$-wave term dropped by us is less than 5% of the 90° $\pi^+$ photoproduction cross-section. This residual, consisting of terms involving the $p$-wave, is given by dispersion relation calculations, and even if these are not entirely correct, the errors thus introduced cannot be significant.

Accordingly we use the theory to tell us what portion of the experimentally observed 90° cross-section is the gauge invariance and recoil $s$-wave. Then this *part* of the $s$-wave is extrapolated in the manner used by BETHE, but including known corrections of the order $\omega/M$.

BENEVENTANO *et al.* ([7]) use an empirically suggested extrapolation to $\omega = 1$. They plot the quantity $A_0^+(\omega)/W(\omega)$ of eq. (8) *vs.* energy and note that over the wide interval in their analysis this ratio is constant. They then assume, on this empirical basis, that this quantity will have the same value at threshold. The validity of this assumption is somewhat substantiated by the experimental point of LEISS *et al.* ([13]). Using this 90° extrapolation to get the threshold value of the $\pi^+$ photoproduction matrix element they obtain the following results:

*a*) $(r)_{\omega=1} = 1.87 \pm 0.13$;

*b*) $R(\omega)$ is a decreasing parameter with increasing energy;

*c*) $f^2 = 0.067$.

Theory predicts that $(r)_{\omega=1} = 1.3$ and that the recoil term $R(\omega)$ should increase rather than decrease with increasing energy. However the BENEVENTANO *et al.* analysis has been supported by data on the $\pi^-/\pi^+$ ratio from deuterium at low energies. The experimental value of 1.45 at 175 MeV seemed to be consistent with the extrapolation to 1.87 at threshold rather than 1.3. Recently BALDIN has made a calculation of the ratio from deuterium taking into account more carefully than has been done before the kinematics of the deuteron, the final interaction and Coulomb corrections which are associated with the different charge states of the final nucleons in the $\pi^-$ and $\pi^+$ production processes. BALDIN finds that experimental results of BENEVENTANO *et al.* from deuterium are consistent with the predictions of theory for free neutrons and free protons ([3,14]).

We therefore feel justified in using the theoretical evaluation of the recoil term in our extrapolation to threshold ([15,16]). It should be noted that the proper inclusion of the direct interaction term $-v^2/2\varkappa^2$, in the 90° cross-section,

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([13]) J. E. LEISS, C. S. ROBINSON and S. PENNER: *Phys. Rev.*, **98**, 201 (1955).

([14]) The deuterium $\pi^-/\pi^+$ ratio experiments and Baldin's analysis are not in complete agreement at slightly higher energies, according to BERNARDINI (private communication), but the extrapolation to 1.87 at threshold does not seem to be justified.

([15]) In adopting our extrapolation we have chosen to ignore the experimental points at very low energy. These points rest critically on an absolute knowledge of the betatron energy calibration. In addition the technique is much less direct than that of the emulsion experiment. The points are ignored because such a procedure turns out to be a rewarding one. The fact that this is so is not taken to mean that the points are necessarily wrong but only that further experiments are needed in this region.

([16]) It should be noted that the slope of this extrapolation in energy at 170 MeV does not match well the apparent slope of the experimental data at that point. This discrepancy, together with those mentioned in (14) warrant further study of both experiments and theory in this energy region.

relieves the recoil term, $-R(\omega)$ of some of the burden assigned to it in the analysis of BENEVENTANO *et al.*, and therefore leads to a lower value of $(r)_{\omega=1}$.

In performing our calculation, we use the experimental data at $\omega = 1.1$, corresponding to $E_\gamma = 170$ MeV. This is the lowest energy used in previous analyses, at which the incident photon energy is well defined by an observation of the meson energy. We extrapolate in accordance with the theoretical cross-section dependence:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} \equiv A(\omega) = \frac{2e^2 f^2}{\mu^2}\frac{W(\omega)}{\varkappa\omega}\left[1 - \frac{v^2}{2\varkappa^2} - \frac{g_\mathrm{p} + g_\mathrm{n}}{M}\omega\left(1 - \frac{v^2}{2}\right) + P'(\omega)\right], \tag{11}$$

where $P'(\omega)$ contributes less than 2% to the magnitude of the cross-section at 170 MeV. It should be noted that the extrapolation is independent of the magnitude of any assumed value for $f^2$.

The transition rate for the $\pi^- \to \gamma$ radiative capture process at $\omega = 1$ may now be written:

$$w(\pi^- + \mathrm{p} \to \mathrm{n} + \gamma) = 4\pi\left[\frac{\mathrm{d}\sigma\,(90°,\, \omega = 1.1)}{\mathrm{d}\Omega}\right]^{\mathrm{exp.}} \frac{A(1)}{A(1.1)}(r)_{\omega=1}\left(\frac{2\varkappa^2}{q^2}\right)_{\omega=1}(v_\pi + v_\mathrm{p})_{\omega=1}c\,,$$

where

$$\left[\frac{\mathrm{d}\sigma(90°,\, \omega = 1.1)}{\mathrm{d}\Omega}\right]^{\mathrm{exp.}} = (5.6 \pm 0.4)\,10^{-30}\ \mathrm{cm^2/sr}\,.$$

$$A(1)/A(1.1) = 2.49\left(\frac{q}{\varkappa}\right)_{\omega=1}$$

$$(r)_{\omega=1} = 1.3\,,$$

$$(v_\pi + v_\mathrm{p})c = 3.45 \cdot 10^{10}\,(q)_{\omega=1}\ \mathrm{cm/s}\,.$$

Therefore

$$w(\pi^- + \mathrm{p} \to \mathrm{n} + \gamma) = 1.48 \cdot 10^{-17}\ \mathrm{cm^3/s}\,. \tag{12}$$

It is also of interest to note that the extrapolated $f^2$ is now 0.073.

## 3. – The extrapolation to threshold of the charge exchange amplitude.

The small isotopic triplet and isotopic singlet scattering amplitudes, $a_3$ and $a_1$, have been extrapolated to threshold by OREAR [5] who assumes that both $a_1 = \alpha_1(\mu/k)$ and $a_3 = \alpha_3(\mu/k)$ are constant for low kinetic energy pions. He finds a best fit for $a_1 - a_3 = 0.27 \pm 0.015$ which agrees well with measurements of charge exchange at 20, 30 and 40 MeV and higher. Of course the validity of the assumption that $a_1 - a_3$ remains accurately constant at this value from 20 MeV to zero energy can be decided in an *a priori* way only for a model of *s*-wave pion nucleon scattering. But at present even the fact that

the difference between the $\pi^+$ and $\pi^-$ elastic amplitudes $a^+$ and $a^-$, which is the charge exchange scattering amplitude, is much larger than their sum $(a^+ - a^-)/(a^+ + a^-) \sim 5$ is at variance with expectations from perturbation theory or expansions in the pion nucleon mass ratio, which suggest that this value should be much smaller than one.

However some theoretical statement can be made about the expected behaviour of the low energy $s$-wave amplitudes which do not depend upon being able to calculate from a model in a detailed way. In describing the possible energy dependence of the low energy amplitude it is convenient to neglect the difference between the laboratory and center of mass energies for the pion nucleon system in first approximation. Then it follows from crossing symmetry that the difference between the real part of the negative and positive scattering amplitudes, even for a single partial wave is an odd function of $\omega = \sqrt{k^2+\mu^2}$ and the sum is even, *i.e.*

$$\frac{\alpha_1 - \alpha_3}{k} \sim \omega f(\omega^2)\,, \tag{13}$$

and

$$\frac{2\alpha_3 + \alpha_1}{k} \sim g(\omega^2)\,. \tag{14}$$

Although $(2\alpha_3 + \alpha_1)/k = \text{constant}$ is compatible with (14), the assumption that $(\alpha_1 - \alpha_3)k$ is independent of $\omega$ is incompatible with (13). (Of course it might be a very accurate approximation in the region between 20 MeV ($\omega = 1.14\ \mu$) and 40 MeV ($\omega = 1.3\ \mu$)). If the low energy charge exchange data are fitted to a « simple » curve restricted by the requirement (13) then the zero kinetic energy extrapolation will differ from the Orear values. The new extrapolated value will generally be lower than the average between $\omega = 1.14\ \mu$ and $1.3\ \mu$, since it must now vanish when $\omega = 0$.

A motivation for a possible $\omega$-dependence of $(\alpha_1 - \alpha_3)\mu/k$ follows from an approximate decomposition of dispersion relations for pion nucleon scattering. If the 33 phase shift $\alpha_{33}$ is taken as the only non zero $p$-wave phaseshift then, in the limit of infinite nucleon mass, the charge exchange $s$-wave scattering satisfies the equation [17],

$$(\alpha_1 - \alpha_3)\frac{\mu}{k} = \left(\frac{\omega}{\mu}\right)\left[6f^2 - \frac{4\mu^2}{\pi}\int_0^\infty \frac{dk'}{\omega' k'^2}\sin^2\alpha_{33}(k') + \right. \tag{15}$$

$$\left. + \frac{2\mu^2}{\pi}\int_0^\infty \frac{dk'}{\omega'^3}(\sin^2\alpha_1 - \sin^2\alpha_3)\right] + \left(\frac{\omega}{\mu}\right)^3\left[\mu^4\frac{2}{\pi}P\int_0^\infty \frac{dk'}{\omega'^3}\frac{\sin^2\alpha_1 - \sin^2\alpha_3}{k'^2 - k^2}\right],$$

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([17]) G. CHEW, M. L. GOLDBERGER, F. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957); cf. also: R. OEHME: *Phys. Rev.*, **10**, 1503 (1955); **102**, 1174 (1956); A. SALAM: *Proceedings CERN Conf.* (1956), p. 176.

where $f^2$ is the renormalized meson nucleon coupling constant $(g(\mu/2M))^2$.

The first integral on the right hand side is much larger than either of the integrals over the $s$-wave phaseshifts. From the evaluation of this integral by CINI and FUBINI [18] we have:

$$(15') \qquad (\alpha_1 - \alpha_2)\frac{\mu}{k} \simeq \left(\frac{\omega}{\mu}\right)[6f^2 - 0.31],$$

For $f^2 \sim 0.09$ then $(\alpha_1 - \alpha_3)\mu/k \sim 0.23(\omega/\mu)$ which agrees with the linear extrapolation in Fig. 1. For smaller $f^2$ ($\sim 0.08$) the initial slope is somewhat less, but then a high (0.27) extrapolated threshold value would be even more remarkable. The bracket multiplying $(\omega/\mu)^3$ is much smaller than that of $(\omega/\mu)$ because of the very small values of $\alpha_1$ and $\alpha_3$ except possibly for very large $q$ where its behaviour is unknown. For small $q$ it is approximately constant. Thus near threshold and below $(\alpha_1 - \alpha_3)\mu/k$ might be expected to be fairly well approximated by a constant times $(\omega/\mu)$. There are two types of corrections: small $\omega^3$ terms such as arise from the second bracket on the right hand side of (15) and kinematic corrections for the transformation to the center of mass system. We first note that $\omega$ should be the total center of mass energy minus the nucleon mass, but this will make only a negligible difference in plotting and fitting low energy data. Secondly there is a multiplication factor $M/(M+\omega)$ where $M$ is the nucleon mass. Therefore we finally take as a possible form to extrapolate low energy charge exchange data

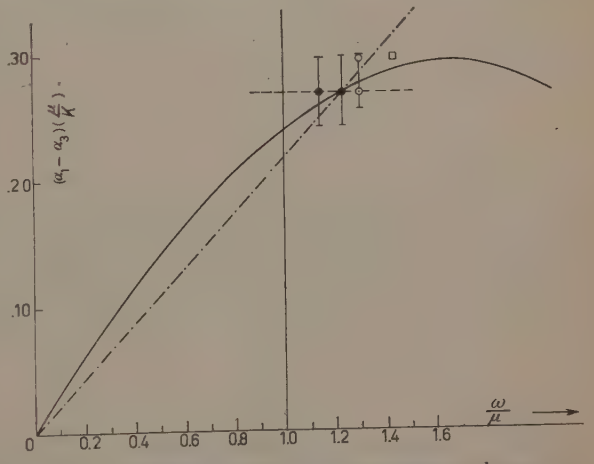


Fig. 1. – Extrapolation of low energy charge exchange data. The ● are from the data of SPRY [21a]. The ○ are from the experiment of TINLOT and ROBERTS [21b]. The ⊙ is the adjusted value of BETHE and DE HOFFMANN [1a] to fit Spry's cross-section measurements. The □ is from the analysis of data of BODANSKY, SACHS and STEINBERGER [22] (no quoted error). The - - - - curve is the Orear extrapolation; ·-·-·-· is the linear extrapolation $(\alpha_1 - \alpha_3)\mu/k = 0.22\,(\omega/\mu)$ and ——— is the curve of eq. (15').

$$(16) \qquad (\alpha_1 - \alpha_3)\frac{\mu}{k} = \frac{1}{1 + (\omega/M)}\left[a\left(\frac{\omega}{\mu}\right) + b\left(\frac{\omega}{\mu}\right)^3\right] \qquad a \gg b\,, \quad a > 0\,.$$

[18] M. CINI and S. FUBINI: *Nuovo Cimento*, **3**, 764 (1956).

According to OREAR $(\alpha_1 - \alpha_3)\mu/k$ remains about constant until 150 MeV kinetic energy and then decreases. In Fig. 1 we plot the curve (16) with $a$ and $b$ adjusted so that the right hand side equals 0.27 at 30 MeV and again at 150 MeV. Then $a = 0.310$, $b = -0.0318$, and the extrapolated threshold value of $(\alpha_1 - \alpha_3)\mu/k = 0.24$ (*).

The amplitude is also plotted with the extrapolation formula $a'(\omega/\mu)$ and the constant approximation (Orear) which give 0.22 and 0.27 respectively. Both eq. (16) and the linear extrapolation give lower values which differ from the Orear value by much more than the probable error based only upon the scattering data. We feel that the uncertainty in the extrapolation procedure is the greatest possible source of error in the threshold value of the charge exchange amplitude which enters into the Panofsky ratio, and that « probably » the present scattering data imply a lower value than 0.27, about 0.24 or perhaps

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(*) One should point out that a change in $(\alpha_1 - \alpha_3)\mu/k$ at threshold from Orear's value to 0.24 has the effect, in the PUPPI-STANGHELLINI [19] fit of dispersion relations for pion nucleon scattering, of lowering both the coupling constants deduced from the $D_+$ and $D_-$ plots of the same amount, say $\lesssim 0.01$. This is not going to change the well known discrepancy between the two coupling constants. However we wish to point out that there is no theoretical reason why

$$(2\alpha_3 + \alpha_1)\frac{\mu}{k} = \frac{3}{2}[D_+(0) + D_-(0)] \simeq -0.06$$

should be a constant independent of $\omega$ as assumed in the Orear extrapolation (cf. eq. (14)).

From dispersion relations it is found in fact that a reasonable expression is

$$(2\alpha_3 + \alpha_1)\frac{\mu}{k} = a + b\omega^2 , \tag{N.1}$$

where, because of the observed very small value of $D_+(0)+D_-(0)$ as opposed to $D_+(0)-D_-(0)$, $a$ is expected to be of the same order of $b$. Now $(2\alpha_3+\alpha_1)$ is usually deduced from the difference of two large numbers with large errors, because experiments give separately $\alpha_3$, $2\alpha_1+\alpha_3$, $\alpha_3-\alpha_1$. What reduces the errors in the conventionally accepted result $-0.06 \pm 0.02$ is just the assumption that $\alpha_1/k$ and $\alpha_3/k$ are constant. On the other hand if one introduces two parameters in (N.1), large uncertainties may arise in the threshold value of $(2\alpha_3+\alpha_1)\mu/k$. We emphasize that if instead of the value $-0.06$ one assumes

$$(2\alpha_3 + \alpha_1)\frac{\mu}{k} = 0.00 ,$$

the $f^2$ deduced from the points in the energy range $1.5 < \omega < 2$ of the $D_-$ plot increases of $\sim 0.02$ and the corresponding one from the $D_+$ plot is lowered by the same amount. This is not inconsistent, in our opinion, with experimental scattering data, due to the uncertainty of the extrapolation (N.1). However if additivity is assumed, the level shifts in $\pi^-$ mesic atoms do confirm the Orear value $-0.06$.

[19] G. PUPPI and A. STANGHELLINI: *Nuovo Cimento*, **5**, 1305 (1957).

even lower. By assuming 0.24 the threshold charge exchange transition rate is easily calculated ([1a])

$$w(\pi^- + \mathrm{p} \to \pi^0 + \mathrm{n}) = \left(1 + \frac{\mu}{M}\right) \frac{8\pi}{9} \frac{(\alpha_1 - \alpha_3)^2}{k^2} v_0 \simeq 2.25 \cdot 10^{-17} \text{ cm}^3/\text{s} . \tag{17}$$

Dividing this value by the extrapolated threshold value of the radiative capture transition rate given by eq. (12) one obtains

$$R = 1.52$$

which agrees, perhaps too well, with the measured value (1).

## 4. – Discussion about the proposed neutral pion of isotopic spin zero.

To state our argument against the existence of the $\pi_0^0$ we start by pointing out that, in the assumption that only $\pi_0^0$'s are produced in the Panofsky experiment, the $s$-wave $\pi_0^0$-nucleon scattering amplitude $a_0/\mu$ in the only possible state $T = \frac{1}{2}$ is immediately determined. In fact, the charge exchange transition rate in this case is obtained simply by replacing $(\sqrt{2}/3)(\alpha_1 - \alpha_3)/k$ with $a_0/\mu$ in the conventional expression for the charge exchange $\pi^-$ capture transition rate ([1a]):

$$w(\pi^- + \mathrm{p} \to \pi_0^0 + \mathrm{n}) = \left(1 + \frac{\mu}{M}\right) 4\pi \frac{a_0^2}{\mu^2} v_0 . \tag{18}$$

In order to determine $a_0$ we use for the left hand side in eq. (18) the experimental value of $R = 1.5$ together with the conventional value for $w(\pi^- + \mathrm{p} \to \gamma + \mathrm{n})$. If we took our value our argument would be strengthened. This leads to

$$a_0^2 = 0.011 . \tag{19}$$

Now let us consider the charge exchange $\pi^- + \mathrm{p}$ experiments at energies up to, say, 40 MeV. In this case both neutral mesons are produced and expected to be indistinguishable in the detection: therefore the experimental total charge exchange cross-section must be the sum of the cross-sections for the production of $\pi_0^0$ and the conventional isotopic spin one $\pi^0$. The cross-section for production is obtained immediately from expression (18) assuming a linear dependence of the phaseshift $\alpha_0$ on the momentum $k$ (and neglecting the difference in the velocities of $\pi^-$ and $\pi^0$)

$$\sigma(\pi^- + \mathrm{p} \to \mathrm{n} + \pi_0^0) = 4\pi \frac{a_0^2}{\mu^2} . \tag{20}$$

On the other hand the cross-section for $\pi^0$ production is given by the conventional expression

$$\sigma(\pi^- + p \to n + \pi^0) = \frac{8\pi}{9}\left[\frac{(\alpha_1 - \alpha_3)^2}{k^2} + \alpha_{33}^2\right], \tag{21}$$

where the phaseshifts $\alpha_1$, $\alpha_3$, $\alpha_{33}$ must now be determined by using only the elastic scattering experiments of charged pions. We take the values given by various authors at the CERN Conference of 1956 ([20]):

$$\begin{cases} a_3 = -0.11 \\ 2a_1 + a_3 = 0.23 \pm 0.03 \\ a_{33} = 0.235 . \end{cases} \tag{22}$$

The total charge exchange cross-section expected if $\pi_0^0$ exists is now easily calculated by means of (19), (20), (21), (22).

In Table I the cross-section for $\pi_0^0$ production ($\sigma(\pi_0^0)$), the cross-section for $\pi^0$ production ($\sigma(\pi^0)$), the total charge exchange cross-section expected ($\sigma_T$), and the experimental results ([21a,b]) are shown at 20, 30, 42 MeV.

TABLE I.

| $E$ MeV | $\sigma(\pi_0^0)$ | $\sigma(\pi^0)$ | $\sigma_T$ | $\sigma_{exp}$ |
|---|---|---|---|---|
| 20 | 3.3 | $4.8 \pm 0.6$ | $8.1 \pm 0.6$ | $5.0 \pm 0.8$ |
| 30 | 3.3 | $5.5 \pm 0.6$ | $8.8 \pm 0.6$ | $5.7 \pm 0.9$ |
| 42 | 3.3 | $6.5 \pm 0.6$ | $9.8 \pm 0.6$ | $6.8 \pm 1.2$ (*a*)<br>$7.9 \pm 1.8$ (*b*) |

It is clear that experimental data seem to be inconsistent with the existence of the $\pi_0^0$, and agree very well with the values calculated from the scattering experiments of charged pions only.

([20]) A. I. MUKHIN *et al.*: *Proc. CERN Conf.* (1956), p. 204; G. FERRARI *et al.*: *Proc. CERN Conf.* (1956), p. 230; M. H. ALSTON *et al.*: *Proc. CERN Conf.*, (1956), p. 236; D. E. NAGLE *et al.*: *Proc. CERN Conf.* (1956), p. 238.

([21]) *a*) W. SPRY: *Phys. Rev.*, **95**, 1295 (1954); *b*) J. TINLOT and A. ROBERTS: *Phys. Rev.*, **95**, 137 (1954).

([22]) D. BODANSKY, A. SACHS and J. STEINBERGER: *Phys. Rev.*, **93**, 1367 (1954).

In addition we only want to point out that one can exclude that in the capture of $\pi^-$ at rest an appreciable fraction of mesons with mass different from $\mu_0 = 264\ m_e$ are produced because in this case one would find for the $\gamma$-ray spectrum a superposition of two rectangular distributions, in contrast with experimental results. This fact rules out the possibility of reducing the discrepancies pointed out above by adjusting the value of $a_0$ taking an admixture of $\pi_0^0$ and $\pi^0$ mesons in the Panofsky experiment.

Specifically, a comparison of $\sigma_{exp}$ and $\sigma(\pi^0)$ in Table I indicates that charge independence and the analyses of $\pi^+$ and $\pi^-$ elastic scattering are compatible with experimental charge exchange measurements only if $\sigma(\pi_0^0) \lesssim 1$ mb, *i.e.* $|a_0| \lesssim 6.3 \cdot 10^{-2}$.

Now the total width of the observed $\gamma$ spectrum in the Panofsky experiment coincides with the width of the spectrum from the lightest neutral pion which is expected to be much larger than the width from the slower (heavier) neutral pion, and very much larger than the shift in the centers of the two $\gamma$-spectra. The experimental measurement yields therefore for the lightest particle the conventional mass value $\mu_0 = 264\ m_e$ and velocity $\beta_0 = 0.2$ c.

On the other hand if one uses the conventional threshold values this measured mass cannot belong to the conventional $\pi^0$ since the calculated Panofsky ratio from it alone is already too large. Therefore the measured Panofsky ratio and the radiative capture rate together with the scattering lengths of both mesons and the mass and velocity of the $\pi_0^0$ determine uniquely the velocity and hence the mass of the $\pi^0$. This turns out to be about $5\ m_e$ greater than the measured mass of the $\pi_0^0$. Furthermore one calculates that the ratio of $\pi^0$ to $\pi_0^0$ in this experiment must be about two, and the $\pi^0$ velocity about 0.5 times the velocity of the $\pi_0^0$. The observed $\gamma$-ray energy spectrum should then consist of two rectangles, one centered at about 67 MeV, and the other one at about 69 MeV. The spectrum corresponding to the $\pi_0^0$ $\gamma$-rays should be twice as wide and have one quarter the height of that corresponding to the $\pi^0$ $\gamma$-rays. The combination gives a spectrum quite different for the observed one.

### *Note added in proof.*

BALDIN and KABIR ([23]) have also suggested that the elastic scattering experiments of charged pions might be consistent with $\alpha_3 \approx \alpha_1$, at variance with the conventional values. In this case the discrepancy between theory and experiment shown by the data of Table I would no longer exist, as $\sigma(\pi^0)$ would be much smaller. We point

([23]) A. M. BALDIN and P. K. KABIR: *Nuovo Cimento*, **9**, 547 (1958).

out, however. that this assumption is inconsistent with dispersion relations. both in the subtracted form ([19]) and in the form connecting directly $(\alpha_3 - \alpha_1)/k$ with $f^2$ and the integrated cross-sections.

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## RIASSUNTO

Si fa vedere che l'apparente inconsistenza tra interazioni di pioni a bassa energia può venire risolta se si fa uso di estrapolazioni suggerite dalla teoria per i processi di scambio carica e di fotoproduzione invece delle solite estrapolazioni lineari.

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# On the Momentum Dependence of the Nuclear Potential (*).

C. Villi

*Istituto di Fisica dell'Università - Trieste*
*Istituto Nazionale di Fisica Nucleare - Sottosezione di Trieste*

(ricevuto il 20 Giugno 1958)

**Summary.** — It is shown that the potential energy of a nucleon in the nucleus, evaluated in the first approximation of the perturbation method or on the basis of Brueckner's theory, obeys a hyperbolic partial differential equation, which is independent of any nuclear parameter and is established by the antisymmetry properties of the nucleon assembly only. This nuclear equation rules the dependence of the potential both on the nucleon momentum and the nuclear density. The particular choice of the two-body forces or of the nucleon-nucleon phaseshifts for the evaluation of the potential energy of the nucleus, implies a specialization of the Cauchy problem, related to this equation, regardless of the saturation prescriptions of nuclear forces. It is shown that there exists a class of solutions of this nuclear equation which cannot be derived, in the first approximation of the perturbation method, from any of the known two-body potentials. This class of solutions, however, leads to the saturation of the nuclear binding energy and density as well as to the experimental value of the symmetry energy and to the correct behavior, in the low energy region, of the real and imaginary parts of the nuclear potential. A mathematical proof is given of the dependence of the potential inside the Fermi sphere on even powers of the nucleon momentum, as required by the invariance prescription of the potential with respect to time reflection. The linear dependence of the potential on the square of the nucleon momentum, and the so called nucleon effective mass approximation, is discussed in the light of the correspondence principle, which has been used to describe the motion of a nucleon in nuclear matter. Finally, it is shown that the Johnson-Teller, Schiff-Thirring and Drell-Huang theories of nuclear saturation implicitly involve only particular solutions of the considered nuclear equation.

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(*) *Note added in proof.* – The author thanks Prof. L. Rosenfeld for illuminating discussions on the subject of this paper and for having communicated some results at the *International Conference on Nuclear Physics* (Paris, July 1958). Some important comments by Prof. V. F. Weisskopf are also gratefully acknowledged.

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## Introduction.

Considerable progress has recently been made towards developing a general theory of the nucleus, based on the interactions between individual nucleons, by relating its properties to the two-body forces. However, although potentially capable of yielding valuable and even decisive information about nuclear forces, the study of heavy nuclei cannot at present give us more than a certain amount of corroborative evidence regarding some general features of the two-nucleon systems inside the nucleus. For this reason it has been thought that an examination of the saturation properties and of some other features of nuclear matter, following a less restrictive point of view, should be of interest. This paper, which has been largely inspired by the refreshing Brueckner's approach to the saturation problem, is an attempt to outline a treatment of the heavy nucleus on the basis of a well defined approximation and according to very general physical and mathematical prescriptions, without resorting to preconceived conjectures regarding the behavior of the nucleons in nuclear matter.

In Sect. **1** it will be shown that the nuclear potential, evaluated in the first approximation of the perturbation method, or, according to Brueckner's theory, in terms of nucleon-nucleon phaseshifts, is a special solution of a hyperbolic differential equation, which rules the dependence of the potential both on the nucleon momentum and the nuclear density. The well known circumstance that the average binding energy per nucleon is equal to the total energy of the most energetic nucleon will be derived in Sect. **2** from the obvious condition that at zero density the nuclear potential should be zero independently of the state of motion of the nucleon. Solutions of the nuclear equation will be determined and briefly discussed in Sect. **4** and applied in Sect. **5** to a study of the saturation of the volume energy and density in agreement with the experimental value of the nucleus' symmetry energy and with the low energy behavior of the real and imaginary parts of the nuclear potential. The dependence of the potential on the square of the nucleon momentum will be derived in Sect. **5** using the correspondence principle, which allows a sound definition of the nucleon effective mass. Finally, in Sect. **6** it will be shown that the Johnson-Teller, Schiff-Thirring and Drell-Huang theories of nuclear saturation involve only special solutions of the considered nuclear equation.

## 1. – The differential equation for the nuclear potential.

Let $V_{0m}^{(n)}$, $\boldsymbol{O}_m^{(n)}$ and $v_m^{(n)}(r_{12})$ be respectively the potential strength, the exchange operator and the radial function of the $m$-th term of the $2n$-th order contribution to a central two-body potential, which will be written in the

following general form

$$V(r_{12}) = \sum_{n=1} V^{(2n)}(r_{12}) = \sum_{n=1} \sum_{m=1} V_{0m}^{(2n)} \boldsymbol{O}_m^{(2n)} v_m^{(2n)}(r_{12}) . \tag{1.1}$$

On calling $\Omega$ the nuclear volume and

$$\psi_i(j) = \Omega^{-\frac{1}{2}} \exp [i\boldsymbol{k}_i \cdot \boldsymbol{r}_j] \chi_i(\sigma_j) \nu_i(r_j) \tag{1.2}$$

the free particle wave functions satisfying boundary periodic conditions, where $\chi_i(\sigma_j)$ and $\nu_i(r_j)$ are respectively the two-row spinors fixing the $j$-th spin and isobaric spin state of the $i$-th particle, the nucleon potential energy, in the first approximation of the perturbation method, is given by

$$\mathcal{V}(k_j, \varkappa) = \sum_{i=1}^{A-1} \int \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 \, \psi_i^*(1) \psi_j^*(2) V(r_{12}) \begin{vmatrix} \psi_i(1) & \psi_i(2) \\ \psi_j(1) & \psi_j(2) \end{vmatrix} , \tag{1.3}$$

where (*) $\varkappa = (9\pi/8)^{\frac{1}{3}} (\mu/\eta)$ is the maximum momentum allowed to the nucleons in the nucleus at the density $A/\Omega = (3/4\pi)(\mu/\eta)^3$. The summation over spin and isobaric spin states gives

$$\sum_{i=1}^{A-1} \psi_i^*(1) \psi_j^*(2) \boldsymbol{O}_m^{(2n)} [\psi_i(1)\psi_j(2) - \psi_i(2)\psi_j(1)] = \tag{1.4}$$

$$= \Omega^{-2} \left\{ a_m^{(2n)} - b_m^{(2n)} \sum_{i=1}^{A-1} \exp [i(\boldsymbol{k}_i - \boldsymbol{k}_j) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)] \right\} ,$$

where the numbers $a_m^{(2n)}$ and $b_m^{(2n)}$ are obviously fixed by the exchange operator $\boldsymbol{O}_m^{(2n)}$. The nuclear model on which Eq. (1.3) is based assumes that the nucleus is composed by an equal number of neutrons and protons and can be pictured as a region of average particle density extending over all space: the tensor interaction averages to zero and for this reason it has been disregarded in the two-body potential (1.1).

On introducting the function

$$f_m^{(2n)}(\theta_i) = -(M/2\pi) V_{0m}^{(2n)} \int \exp [i(\boldsymbol{k}_i - \boldsymbol{k}_j) \cdot \boldsymbol{r}_{12}] v_m^{(2n)}(r_{12}) \, \mathrm{d}\boldsymbol{r}_{12} , \tag{1.5}$$

where $M$ is the nucleon mass and $\theta_i$ is the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_j = \boldsymbol{k}$, Eq. (1.3) becomes

$$\mathcal{V}(k, \varkappa) = -\frac{4\varkappa^3}{3\pi M} \sum_{i=1}^{A-1} \sum_{n,m} \{ a_n^{(2n)} f_m^{(2n)}(0) - b_m^{(2n)} f_m^{(2n)}(\theta) \} . \tag{1.6}$$

(*) A system of units is used where $\hbar = c = 1$ ($\mu^{-1} = 1.4$ fermis).

A straightforward calculation finally gives

$$(1.7)\quad \mathcal{V}(k,\varkappa)=\frac{2\varkappa^3}{3\pi^2}\sum_{n,m} V_{\cup m}^{(2n)}\left\{a_{.n}^{(2n)}\int v_m^{(2n)}(r_{12})\mathrm{d}\boldsymbol{r}_{12}-\frac{3\pi}{\varkappa}\,b_{n}^{(2n)}\int r_{12}j_0(kr_{12})j_1(\varkappa r_{12})v_m^{(2n)}(r_{12})\,\mathrm{d}r_{12}\right\},$$

where $j_1(x)$ are Bessel functions [1]. The lower limit of the radial integral in Eq. (1.7) is equal to the core radius $r_c$ or equal to zero according to whether the repulsive core interaction is assumed to characterize, or not, the behavior at small distances of the two-nucleon potential (1.1). It follows that in theformer case the pairwise correlation of the nucleons is accounted for in Eq. (1.3) simply by a step function $\Theta(r_{12})=1$ for $r_{12}\geqslant r_c$ and $\Theta(r_{12})=0$ for $r_{12}<r_c$. The following discussion, however, is valid for any kind of correlation function.

By noting that the function $f_m^{(2n)}(\theta_i)$, defined in Eq. (1.5), is nothing but the scattering amplitude in Born approximation, Eq. (1.6) by itself suggests that the first order perturbation calculation of the nucleon potential energy can be improved by properly replacing the Born scattering amplitudes with the scattering amplitudes expressed in terms of nucleon-nucleon phaseshifts. In this case, following Brueckner's procedure [2], one obtains

$$(1.8a)\qquad k\leqslant\varkappa\,,$$

$$\mathcal{V}(k,\varkappa)=-\frac{1}{\pi M}\left\{4\int_0^{\frac{1}{2}(\varkappa+k)}F(k')k'^2\,\mathrm{d}k'+\frac{1}{2k}\int_{\frac{1}{2}(\varkappa-k)}^{\frac{1}{2}(\varkappa+k)}F(k')[\varkappa^2-(2k'-k)^2]k'\,\mathrm{d}k'\right\},$$

$$(1.8b)\qquad k\geqslant\varkappa\,,$$

$$\mathcal{V}(k,\varkappa)=-\frac{1}{2\pi Mk}\int_{\frac{1}{2}(k-\varkappa)}^{\frac{1}{2}(k+\varkappa)}F(k')[\varkappa^2-(2k'-k)^2]k'\,\mathrm{d}k'\,,$$

where the function $F(k')$ is expressed as a sum of the scattering amplitudes classified according to spin and isobaric spin substates [3]

$$(1.9)\qquad F(k')=F_{ss}+3F_{ts}+3F_{st}+9F_{tt}\,.$$

Although Eqs. (1.7) and (1.8) are widely different, they nevertheless are solutions of a common mathematical problem underlying the perturbation

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[1] L. Schiff: *Quantum Mechanics* (New York, 1949), p. 77.

[2] K. A. Brueckner, C. A. Levinson and H. M. Mahmud: *Phys. Rev.*, **95**, 219 (1954).

[3] K. A. Brueckner: *Phys. Rev.*, **96**, 508 (1954); B. S. DeWitt: *Phys. Rev.*, **103**, 1565 (1956); N. Fukuda and R. G. Newton: *Phys. Rev.*, **103**, 1558 (1956).

calculation. In fact, taking advantage of the following relations

$$\varkappa \frac{\partial j_1(\varkappa r_{12})}{\partial \varkappa} = -2j_1(\varkappa r_{12}) + \varkappa r_{12} j_0(\varkappa r_{12}), \tag{1.10a}$$

$$\frac{k}{r_{12}} \frac{\partial^2 j_0(k r_{12})}{\partial k^2} = 2j_1(k r_{12}) - k r_{12} j_0(k r_{12}), \tag{1.10b}$$

it is readily established that the nuclear potential (1.7), as well as the potential (1.8), satisfy the following partial differential equation

$$\left(\frac{\partial^2}{\partial k^2} + \frac{2}{k}\frac{\partial}{\partial k} - \frac{\partial^2}{\partial \varkappa^2} + \frac{2}{\varkappa}\frac{\partial}{\partial \varkappa}\right) \mathcal{V}(k, \varkappa) = 0, \tag{1.11}$$

whose hyperbolic nature can be exhibited in normal form using the transformation

$$u = \frac{\varkappa + k}{2}, \qquad v = \frac{\varkappa - k}{2}. \tag{1.12}$$

The close relation existing between the nucleon potential energies (1.7) and (1.8) should be apparent by now since Eq. (1.11) represents the common mathematical background of the perturbation calculation, dependent on a preconceived two-nucleon potential, and the scattering picture, based on the knowledge of the nucleon-nucleon phaseshifts. The choice of the two-body potential in Eq. (1.3) or the introduction in Eq. (1.6) of the scattering amplitudes in terms of phaseshifts, means seeking a solution of Eq. (1.11), consistent with a particular specification of the associated Cauchy problem. Since Eq. (1.11) is the most general expression of that nuclear model which has led so far only to the particular solutions (1.7) and (1.8), the problem arises as to whether it is possible to discover a class of solutions satisfying boundary conditions which are consistent with the saturation of the nuclear binding energy and density, irrespective of any particular choice of the two-nucleon interaction inside the nucleus. If this is found to be possible, one may be tempted to conclude that the failure of the many attempts to account for saturation using Eq. (1.3) depends rather on the two-nucleon potential, used therein for computing $\mathcal{V}(k, \varkappa)$, than on the claimed insufficiency of the first order perturbation calculation. In the contrary case, one would be led to the conclusion that the first approximation of the perturbation method is incapable of accounting for the known properties of nuclear matter independently of the choice of the two-nucleon potential or, more drastically, that the considered nuclear model, far from being euphemistically crude, is rather wrong. Of course, the latter conclusion would require a proof that Eq. (1.11) expresses

the result of the perturbation calculation extended to any order of approximation (*).

The line of thought for determining from a « bad » solution like (1.3) the general equation (1.11) and then looking for those solutions of this equation which fit the physical aspects of the saturation problem as closely as possible, may appear at first rather unconventional. However, it has to be noted that the improvement of the perturbation calculation, achieved by replacing the scattering amplitudes in Born approximation with those expressed in terms of phaseshifts, is in this way taken a step further, because the phaseshifts themselves are also excluded from the theoretical picture and the choice of the solutions of Eq. (1.11) is therefore entirely confined to known experimental facts, without resorting to any preconceived physical assumption concerning the two-nucleon interactions in the interior of the nucleus. The possibility of considering the problem from this, somewhat unusual, point of view is strictly dependent on the fact that the mathematical structure of Eq. (1.11) is connected only to the antisymmetry properties of the nucleon assembly, so that it ultimately expresses, in a compact differential form, the effect of the Pauli principle on the motion of a nucleon in an infinite sea of nuclear matter.

## 2. – The integral equation for the nuclear potential on the characteristic line $k = \varkappa$. The nucleon average volume energy.

An important Theorem, which—to the author's knowledge—has never been clearly stressed hitherto (+), can be derived from Eq. (1.11) by noting that, in the limiting case of zero nuclear density ($\eta = \infty$), the nuclear potential must be zero, independently of the state of motion of the nucleons. It follows that it must be zero also when $k = 0$, *i.e.*

$$\mathcal{V}(0, 0) = 0 . \tag{2.1}$$

We shall now prove that this obvious condition, which of course is true also independently of the finite range of nuclear forces, restricts the class of solutions of Eq. (1.11), having physical interest, to those satisfying the fol-

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(*) *Note added in proof.* – While this paper was in press, this proof has been achieved.

(+) *Note added in proof.* – After the completion of this paper it appeared a paper by N. M. Hugenholtz and L. Van Hove (*Physica*, **24**, 363 (1958)), where this Theorem has been proved along different lines. The Hugenholtz and Van Hove Theorem will be discussed from our standpoint in a subsequent paper to be published in *Nucl. Phys.* (C. Villi: *On a Theorem on the single particle energy in a Fermi gas with interaction*).

lowing integral equation

$$\varkappa^2 \mathcal{V}(\varkappa, \varkappa) = \int_0^{\varkappa} k^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} \, \mathrm{d}k \,, \tag{2.2}$$

valid along the characteristic line $k = \varkappa$. Let us write Eq. (1.11) in the following form

$$\frac{\partial}{\partial k} \left\{ \left( \frac{k}{\varkappa} \right)^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial k} \right\} = \frac{\partial}{\partial \varkappa} \left\{ \left( \frac{k}{\varkappa} \right)^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} \right\} , \tag{2.3}$$

and consider the function $U(k)$ defined as

$$U(\varkappa) = \varkappa^2 \mathcal{V}(\varkappa, \varkappa) - \int_0^{\varkappa} k^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} \, \mathrm{d}k \,. \tag{2.4}$$

Eq. (2.1) simply becomes

$$U(\varkappa) = 0 \,. \tag{2.5}$$

Taking into account the following relation existing among the derivatives of the nuclear potential

$$\frac{\mathrm{d}\mathcal{V}(\varkappa, \varkappa)}{\mathrm{d}\varkappa} = \left\{ \frac{\partial \mathcal{V}(k, \varkappa)}{\partial k} \right\}_{k=\varkappa} + \left\{ \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} \right\}_{k=\varkappa} , \tag{2.6}$$

from Eqs. (2.3) and (2.4) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varkappa} \left\{ \frac{U(\varkappa)}{\varkappa^2} \right\} = \left\{ \frac{\partial \mathcal{V}(k, \varkappa)}{\partial k} \right\}_{k=\varkappa} - \left\{ \left( \frac{k}{\varkappa} \right)^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial k} \right\}_{k=0}^{k=\varkappa} = 0 \,, \tag{2.7}$$

*i.e.* for any solution of Eq. (1.11) it must be $U(k) = ck^2$ ($c = \text{const}$). Then, the condition (2.5) is equivalent to $c = 0$. Replacing $k$ with $\varkappa z$ in Eq. (2.4), we have

$$\mathcal{V}(\varkappa, \varkappa) - \varkappa \int_0^1 z^2 \frac{\partial \mathcal{V}(\varkappa z, \varkappa)}{\partial \varkappa} \, \mathrm{d}z = c \,, \tag{2.8}$$

which at the limit $\varkappa = 0$ reduces to $\mathcal{V}(0, 0) = c$. Thus, the validity of Eq. (2.2) ($c = 0$) implies the condition (2.1). Of course, Eq. (2.2) is satisfied both by the potential (1.7) and (1.8). In the former case the check is straightforward, while in the latter one the function

$$k^2 \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} = -\frac{k\varkappa}{\pi M} \int_{\frac{1}{2}(\varkappa - k)}^{\frac{1}{2}(\varkappa + k)} F(k') k' \, \mathrm{d}k' \,, \tag{2.9}$$

should be integrated over $k$ from 0 to $\varkappa$, using the following rule

$$\int_0^{\varkappa} \mathrm{d}k \int_{\frac{1}{2}(\varkappa-k)}^{\frac{1}{2}(\varkappa+k)} \mathrm{d}k' \to \int_0^{\varkappa/2} \mathrm{d}k' \int_{\varkappa-2k'}^{\varkappa} \mathrm{d}k + \int_{\varkappa/2}^{\varkappa} \mathrm{d}k' \int_{2k'-k}^{\varkappa} \mathrm{d}k \,. \tag{2.10}$$

Eq. (2.2) holds for any nuclear density and therefore also at the nuclear density minimizing the total energy of the nucleus, where the variable $\varkappa$ assumes the values $k_F$, characteristic of the limiting Fermi momentum at that density ($\eta = \eta_F$). It is then easy to show that from Eq. (2.2) follows the well known, but nevertheless surprising, result that the average volume energy per nucleon is equal to the total energy of the most energetic nucleon. Let $\mathcal{E}(k, \varkappa)$, $T(\varkappa)$ and $V(\varkappa)$ be respectively the total nucleon energy and the kinetic and potential energies of the nucleus

$$\mathcal{E}(k, \varkappa) = \frac{k^2}{2M} + \mathcal{V}(k, \varkappa) \,, \tag{2.11}$$

$$T(\varkappa) = \frac{1}{2M} \sum_{j=1}^{A} k_j^2 \,, \tag{2.12}$$

$$V(\varkappa) = \tfrac{1}{2} \sum_{j=1}^{A} \mathcal{V}(k_j, \varkappa) \,. \tag{2.13}$$

The average total energy of the nucleus reads

$$\langle W(\varkappa) \rangle = \frac{3A}{\varkappa^3} \int_0^{\varkappa} k^2 \left\{ \frac{k^2}{2M} + \tfrac{1}{2} \mathcal{V}(k, \varkappa) \right\} \mathrm{d}k \,. \tag{2.14}$$

The necessary condition for $\langle W(\varkappa) \rangle$ to be a minimum at $\varkappa = k_F$ requires that the nuclear potential satisfies the following equation

$$\frac{2}{5} \frac{k_F^2}{M} + \mathcal{V}(k_F, k_F) = \frac{1}{k_F^3} \int_0^{k_F} k^2 \left\{ 3\mathcal{V}(k, k_F) - k_F \left[ \frac{\partial \mathcal{V}(k, \varkappa)}{\partial \varkappa} \right]_{\varkappa = k_F} \right\} \mathrm{d}k \,, \tag{2.15}$$

and therefore the minimum of $\langle W(\varkappa) \rangle$ reads

$$\frac{\langle W(k_F) \rangle}{A} = \frac{k_F^2}{2M} + \tfrac{1}{2} \left\{ \mathcal{V}(k_F, k_F) + \frac{1}{k_F^3} \int_0^{k_F} k^2 \left[ \frac{\partial \mathcal{V}(k, \varkappa)}{\partial k} \right]_{\varkappa = k_F} \mathrm{d}k \right\} . \tag{2.16}$$

On calling $B_V = \langle W(k_F) \rangle / A$ the nucleon average volume energy and taking into account Eq. (2.2), evaluated at $\varkappa = k_F$, Eq. (2.16) simply becomes

$$B_V = E_F + \mathcal{V}(k_F, k_F) \,, \tag{2.17a}$$

where $E_F = k_F^2 / 2M$ is the Fermi energy. Eq. (2.15) reduces to

$$\frac{2}{5} E_F = \frac{\langle V(k_F) \rangle}{A} - \mathcal{V}(k_F, k_F) \,, \tag{2.17b}$$

Therefore, the result $\mathcal{E}(k_F, k_F) = B_V$, shown by Eq. (2.17*a*), follows as a necessary consequence of the physically obvious condition stated in Eq. (2.1).

Eqs. (2.17) are modified by repulsive core boundary conditions, which alter the average kinetic energy of non-interacting point nucleons. On calling $a_c$ a constant proportional to the core radius $r_c$, and taking into account the Lenz correction (4), one has

$$\langle T(\varkappa)\rangle = \frac{3A}{5}\left(\frac{\varkappa^2}{2M}\right)(1 + a_c\varkappa)\,, \tag{2.18}$$

and the saturation conditions become

$$B_V = E_F\left(1 + \frac{6}{5}a_c k_F\right) + \mathcal{V}(k_F, k_F)\,, \tag{2.19a}$$

$$\frac{2}{5}E_F\left(1 + \frac{3}{2}a_c k_F\right) = \frac{\langle V(k_F)\rangle}{A} - \mathcal{V}(k_F, k_F)\,. \tag{2.19b}$$

The preceding relations show that it is not sufficient to be guided only by the experimental features of the nuclear saturation in making the mathematical problem of solving Eq. (1.11) fit the physical aspects of the nucleus as closely as possible since the saturation conditions fix the value of the functions $\mathcal{V}(k, \varkappa)$ and $\langle V(\varkappa)\rangle$ only at the point $k = \varkappa = k_F$, lying on a characteristic curve of Eq. (1.11). It follows that in order to test the physical reliability of the nuclear potential and that of the underlying ideas on which its evaluation has been based, other experimental features of the nuclear matter should be taken into account. It is in fact clear that, while a nuclear potential which does not account for saturation is certainly wrong, the ones which fullfil the stability conditions of the nucleus may not be necessarily correct. For this reason we shall systematically examine the capability of the solutions of Eq. (1.11) to satisfy the saturation prescriptions in agreement with other gross features of the nuclear matter such as the symmetry energy and the energy dependence of the real and imaginary part of the nuclear potential.

## 3. – Solutions of Eq. (1.11) consistent with nuclear saturation.

We shall solve Eq. (1.11) by expanding $\mathcal{V}(k, \varkappa)$ in a power series of the nucleon momentum $k$

$$\mathcal{V}(k, \varkappa) = \sum_{\lambda=0} v_\lambda(\varkappa)k^\lambda\,, \tag{3.1}$$

(4) S. D. Drell and K. Huang: *Phys. Rev.*, **91**, 1527 (1953).

where $v_\lambda(\varkappa)$ are unknown functions of the limiting momentum $\varkappa$. Using the expansion (3.1), Eq. (1.11) becomes

$$\sum_{\lambda=2}\lambda(\lambda-1)v_\lambda(\varkappa)k^{\lambda-2}+2\sum_{\lambda=1}\lambda v_\lambda(\varkappa)k^{\lambda-2}=\sum_{\lambda=0}\left\{v''_\lambda(\varkappa)-\frac{2}{\varkappa}v'_\lambda(\varkappa)\right\}k^\lambda\,, \tag{3.2}$$

where $v'_\lambda=\mathrm{d}v_\lambda/\mathrm{d}\varkappa$, etc. Replacing $\lambda-2$ with $\lambda$ in the first and second summation on the left-hand side of Eq. (3.2), one has

$$(2/k)v_1(\varkappa)+\sum_{\lambda=0}\{(\lambda+2)(\lambda+3)v_{\lambda+2}(\varkappa)-v''_\lambda(\varkappa)+(2/\varkappa)v'_\lambda(\varkappa)\}k^\lambda=0\,. \tag{3.3}$$

It follows that it must be

$$v_1(\varkappa)=0\,, \tag{3.4}$$

$$(\lambda+2)(\lambda+3)v_{\lambda+2}(\varkappa)=v''_\lambda(\varkappa)-(2/\varkappa)v'_\lambda(\varkappa)\,. \tag{3.5}$$

From Eq. (3.4) it follows that no odd powers of the nucleon momentum $k$ appear in the expansion (3.1), as is also required by the invariance of the potential with respect to time reflection [5]. Eq. (3.1) becomes

$$\mathcal{V}(k,\varkappa)=\sum_{\lambda=0}\omega_\lambda(\varkappa)k^{2\lambda}\,, \tag{3.6}$$

where the unknown functions $\omega_\lambda(\varkappa)$ satisfy the equations

$$(2\lambda+2)(2\lambda+3)\omega_{\lambda+1}(\varkappa)=\omega''_\lambda(\varkappa)-(2/\varkappa)\omega'_\lambda(\varkappa)\,. \tag{3.7}$$

It has been numerically proved [6] that, up to terms corresponding to $\lambda=2$, the expansion (3.6), evaluated at normal nuclear density ($\eta_F=1$), reproduces fairly well the momentum dependence of $\mathcal{V}(k,k_F)$, computed either from Eq. (1.7) or (1.8). We shall now extend the validity of this expansion to any nuclear density and assume that it can be cut off regardless of the value of the limiting momentum $\varkappa$. This procedure reveals a class of solutions of Eq. (1.11) which depend on the limiting momentum $\varkappa$ in the most promising manner for obtaining saturation. These solutions, in fact, allow one to overcome the well known difficulty that any two-nucleon potential sufficiently strong to give the correct volume energy causes the nucleus to collapse to a much smaller radius than observed, whereas a potential which accounts for stability at approximately the observed radius does not give an appreciable volume energy.

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[5] L. Eisenbud and E. P. Wigner: *Proc. Nat. Ac.*, **27**, 281 (1941).
[6] E. Clementel and C. Villi: *Nuovo Cimento*, **9**, 950 (1958).

Let $N$ be the maximum value of $\lambda$ considered in the expansion (3.6). This means that all terms with $\lambda > N$ are assumed to be zero, *i.e.* $\omega_{N+s}^{(N)}(\varkappa) = 0$ for $s = 1, 2, \ldots$. Then, the determination of the functions $\omega_\lambda(\varkappa)$ can be carried out by solving the following system of differential equations

$$
(3.8)\qquad \begin{cases} 6\omega_1^{(N)}(\varkappa) = \omega_0''^{(N)}(\varkappa) - \dfrac{2}{\varkappa}\,\omega_0'^{(N)}(\varkappa)\,, \\ \qquad\qquad \ldots\ldots\ldots\ldots \\ 2N(2N+1)\omega_N^{(N)}(\varkappa) = \omega_{N-1}''^{(N)}(\varkappa) - \dfrac{2}{\varkappa}\,\omega_{N-1}'^{(N)}(\varkappa)\,, \\ 0 = \omega_N''^{(N)}(\varkappa) - \dfrac{2}{\varkappa}\,\omega_N'^{(N)}(\varkappa)\,. \end{cases}
$$

Indicating with $C_{2N-1}^{(N)}$ the $(2N+1)$ arbitrary integration constants, the solution of the system (3.8) is found to be

$$
(3.9)\qquad \begin{cases} \omega_N^{(N)}(\varkappa) = C_{2N+1}^{(N)}\varkappa^3 + C_{2N}^{(N)}\,, \\ \omega_{N-1}^{(N)}(\varkappa) = N(2N+1)\left\{C_{2N+1}^{(N)}\dfrac{\varkappa^5}{5} - C_{2N}^{(N)}\varkappa^2\right\} + C_{2N-1}^{(N)}\varkappa^3 + C_{2N-2}^{(N)}\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}
$$

Since we require that the solutions of Eq. (1.11) satisfy also Eq. (2.1), it is clear that the arbitrary constant not multiplied by $\varkappa$, appearing in $\omega_0^{(N)}(\varkappa)$, must be zero.

The necessary, but not sufficient, condition for saturation is that the non zero coefficient of the highest power of $\varkappa$ satisfies at least one of the two inequalities

$$
(3.10a)\qquad C_{2(N-s)+1}^{(N)} > 0\,,
$$

$$
(3.10b)\qquad C_{2(N-s+1)}^{(N)} < 0\,,
$$

where $s = 0, 1, \ldots, (N-1)$. Either of these conditions imply that the potential energy of the nucleus constructed with $\mathcal{V}(k, \varkappa)$, given by Eq. (3.7),

$$
(3.11)\qquad \langle V^{(N)}(\varkappa)\rangle = \frac{3A}{2}\sum_{\lambda=0}^{N}\frac{\omega_\lambda^{(N)}(\varkappa)\varkappa^{2\lambda}}{2\lambda+3}\,,
$$

becomes positive for large nuclear densities ($\eta \ll 1$), thus preventing nuclear collapse. Furthermore, on calling $bC_2^{(N)}$ the coefficient of the term in $\varkappa^2$ appearing in Eq. (3.11), it must be

$$
(3.12)\qquad C_2^{(N)} \geqslant (3/10bM)(1 + a_c k_F)\,.
$$

The trivial solution of Eq. (1.11) is obtained from Eqs. (3.9) assuming all the $C_{2N+1}^{(N)}$'s to be zero except $C_1^{(N)}$. This solution represents the momentum independent square well potential $\mathcal{V}(k, \varkappa) = C_1^{(N)} \varkappa^3$. On calling

$$\omega_0^{(N)}(\varkappa) \equiv \mathcal{V}_0^{(N)}(\varkappa) , \tag{3.13}$$

from Eq. (2.13) one has $\langle V(k_F) \rangle = (1/2) A \, \mathcal{V}_0^{(N)}(k_F)$; Eq. (2.17$b$) gives $\mathcal{V}_0^{(N)}(k_F) = -(4/5) E_F$ and finally from Eq. (2.17$a$) it is obtained

$$B_V = E_F/5 . \tag{3.14}$$

No further comments are needed to conclude that the momentum independent square well potential, although consistent with Eq. (1.11), must be rejected on physical grounds since it is in conflict with nuclear stability.

For a better understanding of the physical implications underlying the mathematical procedure adopted for determining the solutions (3.6) and (3.9) of Eq. (1.11), let us examine the nuclear potential obtained from Eq. (1.1) assuming only terms with $n = m = 1$ and taking $V_{01}^{(2)} = g^2/3$, $\boldsymbol{O}_1^{(2)} = (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})$ and $v_1^{(2)}(r_{12}) = \exp[-\mu r_{12})]/r_{12}$. In this case one has $a_1^{(-)} = 0$ and $b_1^{(2)} = 9/4$, and from Eq. (1.7) it is obtained [7]

$$\mathcal{V}(k, \varkappa) = -\frac{3}{\pi} g^2 \left\{ \varkappa + \frac{\mu^2 + \varkappa^2 - k^2}{4k} \ln \frac{\mu^2 + (\varkappa + k)^2}{\mu^2 + (\varkappa - k)^2} - \right. \\ \left. - \mu \operatorname{arctg} \frac{\varkappa + k}{\mu} - \mu \operatorname{arctg} \frac{\varkappa - k}{\mu} \right\} . \tag{3.15}$$

The potential energy of the nucleus, computed from Eq. (2.13) and (3.15), becomes

$$\frac{\langle V(\varkappa) \rangle}{A} = -\frac{9}{4\pi} g^2 \left\{ \varkappa - \frac{\mu^2}{6\varkappa} + \frac{\mu^2}{2\varkappa} \left( 1 + \frac{\mu^2}{12\varkappa^2} \right) \ln \frac{\mu^2 + 4\varkappa^2}{\mu^2} - \frac{4\mu}{3} \operatorname{arctg} \frac{2\varkappa}{\mu} \right\} . \tag{3.16}$$

Taking into account that for $N = 1$ the nuclear potential coefficients (3.9) are given by

$$\mathcal{V}_0^{(1)}(\varkappa) = \tfrac{8}{5} C_3^{(1)} \varkappa^5 - 3 C_2^{(1)} \varkappa^2 + C_1^{(1)} \varkappa^3 , \tag{3.17$a$}$$

$$\omega_1^{(1)}(\varkappa) = C_3^{(1)} \varkappa^3 + C_2^{(1)} , \tag{3.17$b$}$$

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(7) A. KIND and C. VILLI: *Nuovo Cimento*, **1**, 749 (1955); see also, H. A. BETHE: *Phys. Rev.*, **103**, 1353 (1956).

so that Eq. (3.11) reads

$$\frac{\langle V^{(1)}(\varkappa)\rangle}{A} = \frac{3}{5} C_3^{(1)} \varkappa^5 - \frac{6}{5} C_2^{(1)} \varkappa^2 + \frac{1}{2} C_1^{(1)} \varkappa^3 , \tag{3.18}$$

it is easily established that, for low nuclear densities ($\eta > 2$), Eq. (3.16) identifies with Eq. (3.18) provided the integration constants $C_s^{(1)}$ ($s = 1, 2, 3$) are chosen as follows

$$C_3^{(1)} = \left(\frac{2}{\pi}\right) g^2 \mu^{-4} , \tag{3.19a}$$

$$C_2^{(1)} = 0 , \tag{3.19b}$$

$$C_1^{(1)} = -\left(\frac{2}{\pi}\right) g^2 \mu^{-2} . \tag{3.19c}$$

Eq. (3.19*b*) shows that the condition (3.12) is not satisfied by the potential energy (3.16). The exact equivalence between Eq. (3.16) and (3.18) for large $\eta$ can be easily understood by noting that for low nuclear densities the average distance $\langle r \rangle$ between the nearest neighbors of the $A$ particles, uniformly distributed within the nucleus, increases ($\langle r \rangle > 2.5$ fermis) so that the influence of the small distance behavior of the two-nucleon interaction is immaterial. The preceding result is therefore more general than it may appear at first, because Eq. (3.18) describes the behavior of the nuclear matter for large internucleon separations (low nuclear densities), where any meson-theoretical potential behaves in a Yukawa-like fashion. The crucial point is that Eqs. (3.9), from which Eq. (3.18) has been derived as a very particular case ($N = 1$), have been assumed to be valid also at high nuclear densities, where the small distance behavior of the two-nucleon interaction becomes determinant for obtaining saturation. It is seen that at high nuclear densities the potential energy (3.16) tends to $-\infty$ linearly with the limiting momentum $\varkappa$, whereas the potential energy of the nucleus ,evaluated according to Eq. (3.11), becomes strongly repulsive. Since Eq. (3.16), as is well known, depends on the limiting momentum $\varkappa$ in a manner which is incompatible with Eqs. (2.17), whereas this does not happen with the potential energy (3.11), one of the simplest conclusions one can draw from this illustrative example is that Eqs. (3.9) imply that the nuclear collapse at high densities is prevented by strong, repulsive forces arising between nucleons at short distances, where their radial dependence markedly deviates from the Yukawa one. A crude estimation of the nuclear density at which $\langle V(\varkappa)\rangle$ becomes repulsive can be done by solving the equation $\lambda\!\!\!^{-}(\varkappa) = r_c$, where the relative wave length of the interacting pair of nucleons is evaluated in terms of the average kinetic energy $\langle T(\varkappa)\rangle/A$. As will be seen in Sect. 6, other physical mechanisms, all substantially equi-

valent and equally good, can be suggested as possible explanations for the dependence of $\langle V(\varkappa)\rangle$ on the limiting momentum $\varkappa$, brought about by Eqs. (3.9). For this reason we have intentionally confined ourselves to consider the saturation problem purely from a mathematical point of view: in fact it is rather hard to avoid distrusting detailed speculations on something like the behavior of the potential energy of the nucleus at high nuclear densities, which is, and for ever will be, beyond the limit of experiment.

## 4. - Solutions of Eq. (1.11) consistent with saturation, nucleus symmetry energy and the low energy behavior of the real and imaginary part of the nuclear potential.

The two saturation conditions (2.19) are not sufficient to determine the nuclear density at which the minimum of $\langle W^{(N)}(\varkappa)\rangle$ occurs. Thus, to fix $\eta_F$ and to test the physical reliability of the expansion (3.6), one needs to take into account some additional experimental facts such as the symmetry energy

$$B_{SYM} = a_3^{(exp)} \frac{(A-2Z)^2}{A}, \tag{4.1}$$

and the zero energy limit on the real part of the nuclear potential $\mathcal{V}_R^{(exp)}(0)$. The former quantity is readily evaluated in terms of the solutions of Eq. (1.11) as the difference between the nucleus total energy (2.14), computed for $Z$ protons and $A-Z$ neutrons, and the total energy of a standard nucleus. Neglecting the slight dependence of the $\omega_\lambda^{(N)}(\varkappa)$'s on $A$ and $Z$, it is found

$$a_3^{(N)}(k_F) = \frac{E_F}{3}\left\{1 + \frac{9a_c k_F}{5} + \frac{1}{2E_F}\sum_{\lambda=0}^{N} \lambda\omega_\lambda^{(N)}(k_F)k_F^{2\lambda}\right\}. \tag{4.2}$$

The real part of the potential, felt in the field of a heavy nucleus by a nucleon of energy $E = K^2/2M$ is obtained by transforming Eq. (3.6) using the optical relation

$$k(E) = K\left\{1 - \mathcal{V}_R(E, k_F)/E\right\}^{\frac{1}{2}}, \tag{4.3}$$

*i.e.*

$$\mathcal{V}_R^{(N)}(E, k_F) = \sum_{\lambda=0}^{N} (2M)^\lambda \omega_\lambda^{(N)}(k_F)[E - \mathcal{V}_R^{(N)}(E, k)_F]^\lambda. \tag{4.4}$$

The saturation conditions (2.19) together with the two equations

$$a_3^{(N)}(k_F) = a_3^{(\exp)}, \tag{4.5a}$$

$$\mathcal{V}_R^{(N)}(0, k_F) = \mathcal{V}_R^{(\exp)}(0), \tag{4.5b}$$

can be satisfied provided $N \leqslant 2$. For $N > 2$ other experimental information should be taken into account when determining the nuclear potential parameters $\omega_\lambda^{(N)}(\varkappa)$. Inspection of Eqs. (2.19) and (4.5) shows that in the $N = 1$ case only two out of the three constants $C_s^{(1)}$ $(s = 1, 2, 3)$ can be computed, and, according to Eqs. (3.11) and (3.18), the only possible simplification is to put either $C_1^{(1)} = 0$ or $C_3^{(1)} = 0$. For $N = 2$ the five arbitrary constants $C_s^{(2)}$ are undetermined and the behavior of the potential energy of the nucleus as a function of the limiting momentum $\varkappa$ *is not* uniquely fixed by the minimum conditions (2.19) and by Eqs. (4.5).

4·1. *The $N = 1$ approximation.* – On calling

$$r_1(\varkappa) = [-2\omega_1^{(1)}(\varkappa)/\omega_0^{(1)}(\varkappa)]^{\frac{1}{2}} \tag{4.6}$$

the nucleon potential energy reads

$$\mathcal{V}^{(1)}(k, \varkappa) = \mathcal{V}_0^{(1)}(\varkappa)[1 - \tfrac{1}{2} r_1^2(\varkappa) k^2]. \tag{4.7}$$

The potential energy of the nucleus is given by Eq. (3.18) and the real part of the potential becomes

$$\mathcal{V}_R^{(1)}(E, k_F) = \gamma_1(k_F)\mathcal{V}_0^{(1)}(k_F) + [1 - \gamma_1(k_F)]E, \tag{4.8}$$

where

$$\gamma_N(k_F) = \{1 + 2M\omega_1^{(N)}(k_F)\}^{-1}. \tag{4.9}$$

The dependence of the nuclear potential coefficients on $k_F$, in agreement with the minimum conditions (2.19), is found to be

$$\mathcal{V}_0^{(1)}(k_F) = -\tfrac{1}{2}\{3E_F(1 + \tfrac{6}{5} a_c k_F) - 7B_V\}, \tag{4.10a}$$

$$k_F^2 \omega_1^{(1)}(k_F) = \tfrac{1}{2}(E_F - 5B_V). \tag{4.10b}$$

These relations show that $\mathcal{V}^{(1)}(k, \varkappa)$ is deeper for slow and shallower for fast

particles. An illustrative example of the saturation problem in this approximation is given in Fig. 1.

In the following discussion we shall abandon the crude picture of the nuclear particles as rigid spheres and neglect the Lenz correction assuming, in agreement with cut-off theories, that the two-nucleon potential becomes repulsive at short distances with finite slope. In this way we try to overcome the difficulties arising from the fact that the core repulsion prevents the particles from moving freely in the infinite sea of nuclear matter as would be required by the independent particle model. Then, the nuclear radius $R=r_0A^{\frac{1}{3}}$, where $r_0=(7/E_F^{\frac{1}{2}})$ fermis, consistent with Eqs. (2.17) and (4.5), is fixed by the equation

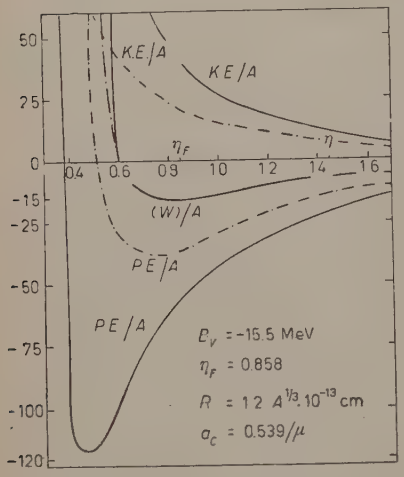


Fig. 1. – Saturation of the nuclear volume energy and density in terms of the potential (4.7). It has been assumed $B_V=-15.5$ MeV and $R=1.2\,A^{\frac{1}{3}}$ fermis. The full line curves have been computed taking into account the effect of the core ($r_c=0.38/\mu$ $a_c=0.539/\mu$). Assuming $C_1^{(1)}=0$, one has $C_3^{(1)}=0.37\,\mu^{-5}$, $C_2^{(1)}=15.8\,\mu^{-2}$ and Eq. (3.18) becomes $\langle V^{(1)}(\eta)\rangle=[1.79\,\eta^{-5}-43.69\,\eta^{-2}]A$. The dotted curves refer to $r_c=a_c=0$. It is found $C_3^{(1)}=0.99\,\mu^{-5}$, $C_2^{(1)}=12.30\,\mu^{-2}$, $C_1^{(1)}=0$ and Eq. (3.18) reads $\langle V^{(1)}(\eta)\rangle=[4.82\,\eta^{-5}-34.11\,\eta^{-2}]A$. The behavior of the total energy of the nucleus is practically unaffected by the Lenz correction for $\eta>0.6$.

$$E_F=\chi-[\chi^2+3a_3^{(\exp)}\mathcal{V}_R^{(\exp)}(0)]^{\frac{1}{2}}, \tag{4.11}$$

$$\chi=\tfrac{21}{10}a_3^{(\exp)}-\tfrac{1}{4}\mathcal{V}_R^{(\exp)}(0). \tag{4.12}$$

It follows that the value of the minimum of the total energy of the nucleus is given by

$$B_V=E_F-\tfrac{21}{5}a_3^{(\exp)}. \tag{4.13}$$

The results following from the $N=1$ approximation are listed in Table I.

Although the present knowledge of the zero energy limit of the real part of the potential is rather scanty, it is generally accepted that $\mathcal{V}_R^{(\exp)}(0)$ has a value included between $-40$ MeV and $-55$ MeV respectively for neutrons [8] and protons [9] elastically scattered by nuclei. Then, from Table I it is seen that Eq. (4.5*b*) is satisfied only for neutrons at $r_0=1.18$ fermis.

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([8]) P. NEMIROWSKY: *Žu. Èksper. Teor. Fiz.*, **30**, 551 (1956); *Dokl. Akad. Nauk SSSR*, **101**, 207 (1955).

([9]) M. A. MELKANOFF, S. A. MOSKOWSKI, J. NODVIK and D. S. SAXON: *Phys. Rev.*, **101**, 507 (1956).

TABLE I. – *Volume energy* $B_V$, *nucleon potential energy at zero momentum* $\mathcal{V}_0^{(1)}(k_F)$ *and the zero energy limit of the real part of the potential* $\mathcal{V}_R^{(1)}(0, k_F)$ (*in* MeV) *evaluated in agreement with the saturation conditions* (2.17) *as functions of the nuclear radius, assuming* $a_3^{(\exp)} = 19.3$ MeV.

| $r_0$ (fermis) | $B_V$ | $\mathcal{V}_0^{(1)}(k_F)$ | $\mathcal{V}_R^{(1)}(0, k_F)$ | $\gamma_1(k_F)$ |
|---|---|---|---|---|
| 0.96 | 6.2 | — 58.1 | — 48.0 | 0.827 |
| 1.04 | — 1.0 | — 71.6 | — 46.0 | 0.642 |
| 1.11 | — 6.9 | — 84.9 | — 43.0 | 0.506 |
| 1.18 | — 11.3 | — 92.4 | — 40.0 | 0.433 |
| 1.23 | — 13.7 | — 96.9 | — 38.0 | 0.392 |
| 1.26 | — 15.5 | — 100.3 | — 36.3 | 0.362 |

The zero energy value $\mathcal{V}_R^{(\exp)}(0) = -55$ MeV cannot be obtained in the $N=1$ approximation without violating either Eqs. (2.17) or Eq. (4.5*a*). The value $\mathcal{V}_R^{(1)}(0, k_F) = -48$ MeV, which is incompatible with nuclear stability (Table I), is the largest possible in this approximation, and corresponds to $E_F = \chi$. With reference to Eq. (4.8) one may note that apparently there exists an energy $E_0 = \gamma_1(\gamma_1 - 1)^{-1}\mathcal{V}_0^{(1)}$ at which $\mathcal{V}_R^{(1)}(E_0, k_F) = 0$, *i.e.* the nuclear matter appears as completely transparent to the incident nucleon waves. At $r_0 = 1.18$ fermis this energy turns out to be $E_0 \sim 83$ MeV. No physical conclusion, however, can be drawn from this circumstance, because Eq. (4.7) is valid only up to nucleon momenta moderately larger than the limiting Fermi momentum $k_F$. A quantitative statement in this regard is obviously connected with the convergence problem of the expansion (3.6), which is difficult to discuss. It can be shown, however, that the solution (3.6) of Eq. (1.11) is valid provided the nucleon momentum satisfies the following inequality

$$\frac{k(E)}{k_F} \ll \left\{\frac{3E_F - 7B_V}{E_F - 5B_V}\right\}^{\frac{1}{2}}, \tag{4.14}$$

which restricts the validity of Eq. (4.8) to low energies only.

4·2. *The* $N=2$ *approximation.* – A more complete description of the nucleus is obtained taking into account also the $k^4$ term in the nucleon potential energy. The nuclear potential coefficients, satisfying Eq. (1.11), are

$$\mathcal{V}_0^{(2)}(\varkappa) = \tfrac{3}{7}C_5^{(2)}\varkappa^7 - 15C_4^{(2)}\varkappa^4 + \tfrac{3}{5}C_3^{(2)}\varkappa^5 - 3C_2^{(2)}\varkappa^2 + C_1^{(2)}\varkappa^3, \tag{4.15a}$$

$$\omega_1^{(-)}(\varkappa) = 2C_2^{(2)}\varkappa^5 - 10C_4^{(2)}\varkappa^2 + C_3^{(2)}\varkappa^3 + C_2^{(2)}, \tag{4.15b}$$

$$\omega_2^{(-)}(\varkappa) = C_5^{(-)}\varkappa^3 + C_4^{(2)}, \tag{4.15c}$$

and their dependence on the density minimizing the total energy of the nucleus

in agreement with the experimental value of the symmetry energy, is given by

$$\mathcal{V}_0^{(2)}(k_F) = \tfrac{1}{8}[43B_V + 36a_3^{(\exp)} - 27E_F] , \tag{4.16a}$$

$$k_F^2\omega_1^{(-)}(k_F) = -\tfrac{1}{4}[35B_V + 60a_3^{(\exp)} - 27E_F] , \tag{4.16b}$$

$$k_F^4\omega_2^{(2)}(k_F) = \tfrac{7}{8}[5B_V + 12a_3^{(\exp)} - 5\ E_F] . \tag{4.16c}$$

The real part of the nuclear potential, following from Eq. (4.4), reads

$$\mathcal{V}_R^{(2)}(E, k_F) = E - E_F\,\varepsilon\,(E, k_F) , \tag{4.17}$$

where

$$\varepsilon(E, k_F) = \xi(k_F)\left\{1 + \frac{2\gamma_2(k_F)[E - \mathcal{V}_0^{(2)}(k_F)]}{\xi(k_F)E_F}\right\}^{\frac{1}{2}} - \xi(k_F) , \tag{4.18}$$

$$\xi^{-1}(k_F) = 2E_F^{-1}\gamma_2(k_F)\omega_2^{(-)}(k_F)k_F^4 . \tag{4.19}$$

Clearly, at the limit $\omega_2^{(2)} = 0$ ($C_5^{(2)} = C_4^{(-)} = 0$) Eq. (4.17) identifies with Eq. (4.8). The nuclear radius, consistent with Eqs. (2.17) and (4.5), is finally obtained by solving the following equation

$$E_F\,\varepsilon\,(0, k_F) = -\mathcal{V}_R^{(\exp)}(0) . \tag{4.20}$$

The validity of the $N = 2$ approximation is bound to much more restrictive conditions than that expressed by the inequality (4.14). The mathematical inspection of this problem shows that Eq. (4.17) can be compared with the experimental energy dependence of the real part of the potential for energies $E \ll E_0$, where $E_0$ is obtained by solving the optical equation

$$\frac{k^2(E_0)}{2M} = E_0 - \mathcal{V}_R^{(2)}(E_0, k_F) , \tag{4.21}$$

being

$$k^2(E_0) = \frac{\{[\omega_1^{(2)}(k_F)]^2 - 4\omega_2^{(2)}(k_F)\mathcal{V}_0^{(2)}(k_F)\}^{\frac{1}{2}} - \omega_1^{(2)}(k_F)}{2\omega_2^{(2)}(k_F)} \tag{4.22}$$

if $\omega_1^{(2)}(k_F) < 0$, $\omega_2^{(2)}(k_F) > 0$ and $\omega_1^{(2)}(k_F) > 2[\omega_2^{(2)}\mathcal{V}_0^{(2)}]^{\frac{1}{2}}$ or $\omega_1^{(2)}(k_F) < 0$ and $\omega_2^{(2)}(k_F) > 0$; and

$$k^2(E_0) = -[2\omega_2^{(2)}(k_F)]^{-1}\omega_1^{(2)}(k_F) \tag{4.23}$$

if $0 \leqslant \omega_1^{(-)}(k_F) \leqslant 2[\omega_1^{(2)}(k_F)\mathcal{V}_0^{(-)}(k_F)]^{\frac{1}{2}}$ and $\omega_2^{(2)}(k_F) < 0$.

The nuclear potential coefficients $\omega_\lambda^{(2)}(k_F)$ and the zero energy limit of the real part of the nuclear potential have been calculated in Table II as functions of the density minimizing the total energy of the nucleus in agreement with the experimental value of the symmetry energy. To solve Eq. (4.20), $\omega_2^{(2)}(k_F)$ must be negative. The value $\mathcal{V}_R^{\text{exp}}(0) = -55$ MeV, obtained from the zero energy extrapolation of the data concerning protons elastically scattered by nuclei ([9]) is found for $r_0 = 1.06$ fermis. At this density, however, Eq. (4.17) becomes meaningless ($E_0 < 0$). It is possible to show that this disappointing situation cannot be overcome by introducing into the model a radial dependence of the Saxon-Wood type. On the contrary, no difficulty is found in order to reproduce, in agreement with saturation and with the experimental value of the nucleus symmetry energy, the real part of the potential felt by incident neutrons in the field of the nucleus. For $r_0 = 1.18$ fermis one has (Table II) $\mathcal{V}_R^{(2)}(0, k_F) = -42.3$ MeV, in agreement with phenomenological analyses of neutron-nucleus elastic scattering. In the neighborhood of this nuclear density Eq. (4.17) can be written as

$$\mathcal{V}_R^{(2)}(E, k_F) = \alpha(k_F)\mathcal{V}_0^{(2)}(k_F) + [1 - \beta(k_F)]E\,, \tag{4.24}$$

where

$$\alpha(k_F) = \frac{\xi(k_F)E_F}{\mathcal{V}_0^{(2)}(k_F)}[1 - \delta(k_F)]\,, \tag{4.25a}$$

$$\beta(k_F) = \gamma_2(k_F)\delta^{-1}(k_F)\,, \tag{4.25b}$$

$$\delta(k_F) = \left\{1 - \frac{2\gamma_2(k_F)\mathcal{V}_0^{(2)}(k_F)}{\xi(k_F)E_F}\right\}^{\frac{1}{2}}. \tag{4.25c}$$

TABLE II. - *Nuclear potential coefficients* $\omega_\lambda^{(2)}(k_F)$ *and zero energy limit of the real part of the nuclear potential* $\mathcal{V}_R^{(2)}(0, k_F)$ *(in* MeV*) computed as functions of* $r_0$ *(in fermis), in agreement with the saturation conditions* (2.17) ($B_V = -15.5$ MeV) *and the experimental value of the symmetry energy* [$a_3^{\text{exp}} = 19.3$ MeV].

| $r_0$ | $\mathcal{V}_0^{(2)}(k_F)$ | $k_F^3\,\omega_1^{(2)}(k_F)$ | $k_F^4\,\omega_2^{(2)}(k_F)$ | $\mathcal{V}_R^{(2)}(0, k_F)$ |
|---|---|---|---|---|
| 1.00 | — 161.8 | 176.9 | — 79.3 | — 69.8 |
| 1.10 | — 133.2 | 119.5 | — 42.4 | — 50.2 |
| 1.20 | — 111.5 | 75.6 | — 13.9 | — 40.4 |
| 1.26 | — 101.5 | 54.7 | 0 | — 36.3 |
| 1.30 | — 94.7 | 42.6 | + 7.9 | — 34.2 |
| 1.40 | — 81.2 | 14.3 | + 25.4 | — 26.8 |

For $r_0 = 1.18$ fermis the potential (4.24) becomes

$$\mathcal{V}_R^{(2)}(E, k_F) = -42.3 \text{ MeV} + 0.5E\,. \tag{4.26}$$

From the condition (4.23) it is found that the validity of Eq. (4.26) is restricted to $E \ll E_0 = 55.9$ MeV. It may be remarked that the linear dependence of $\mathcal{V}_R(E, k_F)$ on $E$ is not only connected with the $N = 1$ approximation but also with the $N = 2$ approximation at nuclear densities for which $\gamma_2(k_F) < < \xi(k_F) E_F \delta^2(k_F)$.

On calling $\sigma_{np}(E) = c_0(E + c_1)^{-1}$ ($c_0 = 8.64$ MeV·barn and $c_1 = 1.08$ MeV) the total neutron-proton cross-section, the imaginary part of the nuclear potential in the low energy region is given by [10]

$$\mathcal{V}_I(E, k_F) = -(5Mc_0/8\pi^2)[E - \mathcal{V}_R(E, k_F)]\Phi(E, k_F)\,, \tag{4.27}$$

where the function $\Phi(E, k_F)$ reads

$$\Phi(E, k_F) = \int^{x_F} \frac{x(1 + x^2 - 2x_F^2)}{[2(1 + x^2 + y)]^{\frac{1}{2}}} \ln \frac{1 + 3x^2 + 2y + 2x[2(1 + x^2 + y)]^{\frac{1}{2}}}{1 + 3x^2 + 2y - 2x[2(1 + x^2 + y)]^{\frac{1}{2}}}\, \mathrm{d}x \tag{4.28}$$

being $x = k/k(E)$, $y = Mc_1/k^2(E)$. The lower limit of integration of Eq. (4.28) is $(2x_F^2 - 1)^{\frac{1}{2}}$ for $E - \mathcal{V}_R(E, k_F) \leqslant 2E_F$, otherwise zero. From Eqs. (4.27) and (4.28) is seen that the imaginary part of the potential $\mathcal{V}_I = \zeta \mathcal{V}_R$ does not become zero for $E = B_F$, but at an energy such that $E' = E_F - \mathcal{V}_R(E', k_F)$. Eq. (4.27) has been evaluated in Table III using the real part of the potential given in Eq. (4.26).

TABLE III. – *Imaginary part of the nuclear potential* (*in* MeV) *evaluated from Eq.* (4.27) *using the real part expressed in Eq.* (4.26) ($5Mc_0/8\pi^2 = 1.322$; $E' = -14$ MeV).

| $E$ (MeV) | $\Phi(E, k_F)$ | $\mathcal{V}_I(E, k_F)$ | $\zeta(E, k_F)$ |
|---|---|---|---|
| 0 | 0.024 | — 1.4 | 0.032 |
| 2 | 0.039 | — 1.7 | 0.041 |
| 4 | 0.036 | — 2.1 | 0.052 |
| 6 | 0.042 | — 2.5 | 0.063 |

Since $\mathcal{V}_I(E, k_F)$, which possesses a maximum at high energies, varies linearly with $E$ in the considered energy interval, the total nuclear potential $\mathcal{V} = \mathcal{V}_R + i\mathcal{V}$ reads

$$\mathcal{V}(E, k_F) = -(42.3 + 1.4i)\ \text{MeV} + (0.5 - 0.2i)E\,. \tag{4.29}$$

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[10] E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **2**, 176 (1955); see also, M. CINI and S. FUBINI: *Nuovo Cimento*, **2**, 75 (1955); A. M. LANE and C. F. WANDEL: *Phys. Rev.*, **99**, 647 (1955); W. B. RIESENFELD and K. M. WATSON: *Phys. Rev.*, **102**, 1157 (1956).

As is well known, in the restricted band of energies where Eq. (4.29) is valid, rather large variations of $r_0$ and $\mathcal{V}_R(E, k_F)$ are permitted by the elastic scattering data provided $r_0[\mathcal{V}_R(0, k_F)]^{\frac{1}{2}}$ is held constant. In our case, the correlation imposed by Eqs. (2.17) and (4.5) between $r_0$ and $\mathcal{V}_R(0, k_F)$ is such that $r_0[\mathcal{V}_R^{(2)}(0, k_F)]^{\frac{1}{2}} = 7.7$ [(MeV)$^{\frac{1}{2}}$·fermi]. It is at present not clear whether this value is in agreement with empirical results adjusted to compensate for smaller nuclear densities, as obtained by other authors. However, the value $r_0 \simeq 1.2$ fermis is in close accord with that derived from measurements of the shift of the $S$ level of mesic atoms ([11]).

It is interesting to note that it is in principle possible to correlate theoretically the capture process of negative muons and the elastic scattering of nucleons by nuclei, so that information on the energy dependence of the real part of the nuclear potential can be also obtained from measurements of the absorption probability in spite of the fact that the muon-nucleus interaction is of electromagnetic nature.

Be $\boldsymbol{p}$, $\boldsymbol{n}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the momenta of the proton, neutron, muon and respectively of the neutrino involved in the capture process $\mathrm{P}+\mu \to \mathrm{N}+\nu$. On calling $B_\mu$ and $M_\mu$ the binding energy and the mass of the muon, the conservation of energy and momentum requires that

$$\mathcal{E}(p, p_F) + B_\mu + M_\mu = \mathcal{E}(n, n) + \nu \,, \tag{4.30a}$$

$$\boldsymbol{p} + \boldsymbol{\mu} = \boldsymbol{n} + \boldsymbol{\nu} \,, \tag{4.30b}$$

where the nucleon total energy is given in Eq. (2.11).

The exclusion principle and the conservation of the energy and momentum restrict the variation of the neutrino momentum between a lower and an upper limit. The latter is obtained from Eq. (4.30$a$) by imposing the condition that the converted neutron falls in an unoccupied state. On calling $B_n = \mathcal{E}(n_F, n_F)$, it is found

$$\nu_{\max}(p) = B_\mu + M_\mu - B_n + \mathcal{E}(p, p_F) \,. \tag{4.31}$$

To evaluate the lower limit $\nu_{\min}(p)$ of the neutrino momentum one should minimize with respect to the angle between $\boldsymbol{p}+\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the function

$$\mathcal{E}(p, p_F) + B_\mu + M_\mu = \mathcal{E}[|\boldsymbol{p} + \boldsymbol{\mu}| + \nu_{\min}(p), n_F] + \nu_{\min}(p) \,. \tag{4.32}$$

It follows that not all protons bound in the nucleus can absorb the muon: in fact, the capture process is forbidden for all protons having a momentum

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([11]) V. L. Fitch and J. Rainwater: *Phys. Rev.*, **92**, 789 (1953); L. M. Cooper and E. M. Henley: *Phys. Rev.*, **92**, 801 (1953).

$p < p_0$, where $p_0$ is evidently obtained by solving the equation

$$\mathcal{E}(p_0, p_F) + B_\mu + M_\mu = B_n + \nu_{\min}(p_0) . \tag{4.33}$$

Finally, the neutron momentum varies from $p_F$ to a maximum momentum $p_{max}$ which can be determined from the equation

$$B_p + B_\mu + M_\mu - \nu_{\min}(p_F) = \mathcal{E}(p_{\max}, p_F) . \tag{4.34}$$

It follows that the converted neutron explores the nuclear potential which is felt by incident nucleons having an energy $E$ included into the interval $0 \leqslant E \leqslant E_0$, where $E_0$ is given by

$$p_{\max} = \{2M[E_0 - \mathcal{V}_R(E_0)]\}^{\frac{1}{2}} . \tag{4.35}$$

Eq. (4.35) discloses an important relationship between the capture process and the elastic scattering of nucleons by nuclei, and clearly suggests a possible role of the negative muon as a probe particle to reveal, in an indirect manner, the dispersive nature of the nuclear potential and not only the charge distribution of nuclei. The momentum dependence of the potential felt by the capturing proton and by the converted neutron strongly influences the lower and upper limit of the neutrino momentum and therefore the value of the absorption probability, evaluated as a sum of contributions of individual particle transitions in which a proton is transformed into a neutron. This happens because the reaction takes place between well defined states of the nucleus and the density of final states is fixed by the density of the neutrino states alone.

It would be interesting to prove whether a momentum dependent potential, whose parameters are determined in agreement with saturation, symmetry energy and the energy dependence of the real and imaginary part of the potential, is also suitable to explain the deviation by a factor 2 or larger from Wheeler's law ([12]) of the measured capture probability in heavy nuclei ([13]) and the values of the mean neutron multiplicities experimentally observed ([14]), which are consistently larger than those calculated on the basis of the conventional statistical model.

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([12]) J. A. WHEELER: *Rev. Mod. Phys.*, **21**, 133 (1949); J. TIOMNO and J. A. WHEELER: *Rev. Mod. Phys.*, **21**, 153 (1949).

([13]) J. W. KEUFFEL, F. B. HARRISON, T. K. K. GODFREY and G. T. REYNOLDS: *Phys. Rev.*, **87**, 942 (1949); A. J. MAYER and J. W. KEUFFEL: *Phys. Rev.*, **90**, 349 (1953).

([14]) M. WIGDOFF: *Phys. Rev.*, **90**, 891 (1953).

**5. – Solutions of Eq. (1.11) derived at $\varkappa = k_F$ from the correspondence principle. The nucleon effective mass.**

For a better understanding of the results obtained in Sect. **4** we shall now attempt to shed light on the solution (3.6) of Eq. (1.11) by noting that for a heavy nucleus the expectation value for the number of mesons present in the same quantum state is considerably larger than 1, so that the wave function for this state approaches the classical potential of nuclear forces. It is then interesting to use the correspondence principle by requiring that the classical relation $\dot{\mathcal{E}}(\boldsymbol{k}, k_F) = \boldsymbol{F}\cdot\boldsymbol{v}$ between the force acting on a nucleon in nuclear matter and its total energy remains valid for the mean values of the quantum theory.

Since the group velocity $\boldsymbol{v}(\boldsymbol{k})$ of the packet and the energy $\mathcal{E}(\boldsymbol{k}, k_F)$ of a nucleon satisfy the following relation

$$\boldsymbol{v}(\boldsymbol{k}) = \nabla_k \mathcal{E}(\boldsymbol{k}, k_F)\,, \tag{5.1}$$

we have

$$\langle \dot{\boldsymbol{v}} \rangle = \langle \nabla_k \boldsymbol{F}\cdot\boldsymbol{v} \rangle = \langle \boldsymbol{F}\cdot\nabla_k \nabla_k \mathcal{E}(\boldsymbol{k}, k_F) \rangle\,. \tag{5.2}$$

Eq. (5.2) corresponds to the classical relation $\dot{\boldsymbol{v}} = \boldsymbol{F}/M$ provided one introduces the following mass tensor

$$\frac{1}{\boldsymbol{M}^*} = \nabla_k \nabla_k \mathcal{E}(\boldsymbol{k}, k_F)\,. \tag{5.3}$$

In the particular case of a spherically symmetric nucleus, the mass tensor (5.3) reduces to the scalar nucleon effective mass

$$M^* = \left\{\frac{\mathrm{d}^2 E(k, k_F)}{\mathrm{d}k^2}\right\}^{-1}, \tag{5.4}$$

*i.e.*, using Eq. (2.4)

$$\frac{M^*}{M} = \left\{1 + M\frac{\mathrm{d}^2 \mathcal{V}(k, k_F)}{\mathrm{d}k^2}\right\}^{-1}. \tag{5.5}$$

Of course, the nucleon effective mass becomes unity for free nucleons or when the nucleon potential energy is arbitrarily assumed to be momentum independent. On the basis of these considerations it is readily established that the modified nuclear particles, having effective mass given by Eq. (5.5), play the role of nucleons in the independent particle model, provided the nucleon potential energy obeys the following differential equation ([6])

$$k^2 \frac{\mathrm{d}^2 \mathcal{V}(k, k_F)}{\mathrm{d}k^2} - 2\mathcal{V}(k, k_F) = -2\mathcal{V}_0(k_F)\,, \tag{5.6}$$

the solution of which, having physical sense, is simply given by the potential (4.7), evaluated at $\varkappa = k_F$. Thus, the linear dependence of the nuclear potential on $k^2$ only can be derived by assuming that the nucleon effective mass, defined according to the correspondence principle, is independent of the nucleon momentum. The correspondence principle is however incapable of fixing the dependence of the potential on the nuclear density and therefore, to study the saturation problem, one must resort to the solutions (3.6) and (3.9) of Eq. (1.11). The parameter defined in Eq. (4.9) for $N = 1$ is nothing but the nucleon effective mass ratio (5.5), which turns out to be independent of the Lenz correction

$$\gamma_1(k_F) = \frac{M^*}{M} = \frac{2E_F}{3E_F - 5B_V}. \tag{5.7}$$

In this connection it has to be remarked that no reliable information on the quantity $M^*$ can be derived simply by examining the low energy behavior of the real part of the nuclear potential, because of the misleading circumstance that Eqs. (4.8) and (4.24) may turn out to be quantitatively identical, although in the latter case the parameter $\alpha(k_F)$, defined in Eq. (4.25*a*), has no relation at all, in the spirit of the correspondence principle, with the nucleon effective mass ratio $\gamma_1(k_F)$ [Eq. (5.7) = Eq. (4.9)]. We point out that the potential (4.26) ($N = 2$) practically identifies with that evaluated by BRUECKNER, EDEN and FRANCIS ([15]), who have implicitely made use of the $N = 1$ approximation of Eq. (1.11). To add to the confusion, it is also worth-while noting that Eq. (4.8) may be successfully used to reproduce at $r_0 = 1.33$ fermis the low energy behavior of protons elastically scattered by nuclei ([9]) ($E \leqslant 5$ MeV), assuming $\gamma_1(k_F) = 0.5$ [$\mathcal{U}_0^{(1)}(k_F) = -110$ MeV]. Although the value $B_V = -20$ MeV following from Eq. (4.10*a*) ($a_c = 0$) may be justified in terms of the rearrangement energy which tends to increase the energy of the most loosely bound nucleon, Eq. (4.10*b*) turns out to be incompatible with the chosen value of $\gamma_1(k_F)$. This contradiction may be shown to be largely independent of the radial behavior of the potential postulated in ref. ([9]), and therefore one has to conclude that the value $M^* = 0.5\,M$, apparently implied by the analysis of MELKANOFF *et al.*, seems to be in conflict, on the basis of the considered nuclear model, with nuclear saturation.

To prove that in practical cases Eq. (4.7), evaluated at $\varkappa = k_F$, reproduces the momentum dependence of the potential evaluated from Eq. (1.7) of (1.8), it is sufficient to check whether the function

$$\mathcal{G}(k, k_F) = \left\{\left(\frac{\partial^2}{\partial \varkappa^2} - \frac{2}{\varkappa}\frac{\partial}{\partial \varkappa}\right)\mathcal{U}(k, \varkappa)\right\}_{\varkappa = k_F}, \tag{5.8}$$

([15]) K. A. BRUECKNER, R. J. FRANCIS and N. C. EDEN: *Phys. Rev.*, **100**, 891 (1955); see also W. E. FRAHN: *Nuovo Cimento*, **4**, 313 (1956).

is independent of the nucleon momentum $k$. In fact, for $\varkappa = k_F$ Eq. (1.11) reduces to the Poisson equation in momentum space

$$\nabla^2 \mathcal{V}(k, k_F) = \mathcal{G}(k, k_F) , \tag{5.9}$$

the solution of which, satisfying the condition that the potential be finite at $k = 0$, identifies with Eq. (4.7) if $\mathcal{G}(k, k_F)$ is independent of $k$ in the interval $0 \leqslant k \leqslant k_F$. Therefore, the deviation of the nucleus behavior from the classical one is simply displayed by a possible momentum dependence of the function $\mathcal{G}(k, k_F)$. For instance, using the non-saturating potential (3.15) it is found

$$\frac{\mathcal{G}(k_F, k_F)}{\mathcal{G}(0, k_F)} = \frac{1}{2}\left(\frac{\mu^2 + k_F^2}{2k_F^2}\right)^2 \left[\ln \frac{\mu^2 + 4k_F^2}{\mu^2} + \frac{3\mu^2 + 8k_F^2}{\mu^2 + 4k_F^2}\right] . \tag{5.10}$$

This ratio varies between 1.015 and 1.073 for $1.2f \leqslant r_0 \leqslant 1.3f$: therefore, within this density interval, the nucleon potential energy (3.15) follows closely Eq. (4.7), where

$$\mathcal{V}_0^{(1)}(k_F) = -\frac{6}{\pi} g^2 [k_F - \mu \operatorname{arctg} (k_F/\mu)] , \tag{5.11a}$$

$$\frac{1}{2} r_1^2(k_F) = \frac{E_F[\gamma_1(k_F) - 1]}{\gamma_1(k_F)\mathcal{V}_0^{(1)}(k_F)} , \tag{5.11b}$$

being the dependence of the nucleon effective mass ratio on the coupling constant $g$ given by

$$\frac{M^*}{M} = \frac{E_F(\mu^2 + k_F^2)^2}{E_F(\mu^2 + k_F^2)^2 + 2g^2\mu^5} . \tag{5.12}$$

## 6. – Solutions of Eq. (1.11) derived from the Johnson-Teller, Schiff-Thirring and Drell-Huang theories of nuclear forces.

It is known that in order to overcome the disappointing results obtained from two-body interaction with regard to nuclear saturation and the apparent incompatibility of the strong, short-range two-body forces between nucleons with the independent particle model of the nucleus, it has been suggested *a*) that the velocity dependence of nuclear forces should be taken into account ([16]); *b*) that the essential features of the independent particle model

([16]) M. H. Johnson and E. Teller: *Phys. Rev.*, **98**, 783 (1955).

for a heavy nucleus in its ground state can be justified on the basis of a non-linear meson theory ([17]) and *c*) that many body forces among nucleons may play an important role in explaining the saturation mechanism of the nucleus ([4]). We shall now prove that all these attempts to study the nucleus ground state along new lines, do not add anything which is not already contained in the simple mathematical form of the nuclear equation (1.11).

The shell-like nuclear potential, postulated by JOHNSON and TELLER ([16]), is a special solution of Eq. (1.11) corresponding to the $N = 1$ case (effective mass approximation). In fact, neglecting the surface term in the nucleus Hamiltonian

$$(6.1)\quad H = \langle W(\varkappa)\rangle = \int \{(2M)^{-1} \sum_j |\nabla\psi_j|^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}\mu_0^2\varphi^2 - g\varphi\sum_j |\psi_j|^2 + \mu_0^{-2} f\varphi \sum_j |\nabla\psi_j|^2\}\, d\boldsymbol{r}\,,$$

and calling $\varphi_0 = g^2\varrho_0\mu^{-2}$ the isotopic singlet in this approximation, the nucleus potential energy reads

$$(6.2)\quad \langle V(\varkappa)\rangle = \left\{\frac{1}{2}\mu_0^2\varphi_0^2 - g\varphi_0\varrho_0 + \frac{3}{5\mu_0^2} f\varphi_0\varrho_0\varkappa^2\right\}\Omega\,.$$

Since $\varrho_0 = A/\Omega = (2/3\pi^2)\varkappa^3$, Eq. (6.2) reduces to $\langle V^{(1)}(\varkappa)\rangle$ given in Eq. (3.18) where

$$(6.3a)\quad C_3^{(1)} = \left(\frac{2}{3\pi^2}\right) f g\mu_0^{-4} = \frac{1}{2k_F^5}(E_F - 5B_V)\,,$$

$$(6.3b)\quad C_2^{(1)} = 0\,,$$

$$(6.3c)\quad C_1^{(1)} = -\left(\frac{2}{3\pi^2}\right) g^2\mu_0^{-2} = -\frac{1}{5k_F^5}(9E_F - 25B_V)\,.$$

Eqs. (6.3) allow the evaluation of the coupling constants $g$ and $f$ in agreement with Eqs. (2.17). Thus, the nucleon potential energy, hidden within the Hamiltonian (6.1) when $(\nabla\varphi)^2 = 0$, is given by Eq. (4.7) and the nucleon effective mass

$$(6.4)\quad \frac{M^*}{M} = (1 + 2Mf\varphi_0\mu_0^{-2})^{-1}\,,$$

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([17]) L. J. SCHIFF: *Phys. Rev.*, **84**, 1 (1951); **86**, 856 (1952); **92**, 766 (1953); W. THIRRING: *Zeits. f. Naturfor.*, **7***a*, 63 (1952); **9**, 804 (1954).

clearly identifies with Eq. (5.7). It may be noted that the *ad hoc* introduction of the new coupling constant $f$ is here implicit in the mathematical procedure of solving the system of differential equations (3.8) in the $N = 1$ approximation [$C_3^{(1)} \sim f$]. Eq. (6.3*a*) fullfils the saturation condition (3.10*a*) ($N = 1$, $s = 0$), whereas Eq. (6.3*b*) conflicts with condition (3.12) and therefore the nucleus Hamiltonian (6.1) exhibits a maximum at low densities. It is evident that the results one can obtain on the basis of the Johnson-Teller theory at the density minimizing the total energy of the nucleus identify with those listed in Table I. These results are not appreciably altered by the radial dependence of the nuclear potential, which is implicit in the nucleus Hamiltonian (6.1). It is, in fact, easy to prove that the Johnson-Teller theory can be thought of as implicitly assuming that the nucleons inside the nucleus interact via two-body forces described by the « equivalent » two-nucleon potential

$$V(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{r}_1, \boldsymbol{r}_2; \varkappa) = g^2(\boldsymbol{k}_i, \boldsymbol{k}_j; \varkappa) \exp[-\mu_0 r_{12}]/\mu_0 r_{12}\,, \tag{6.5}$$

where the strength $g^2(\boldsymbol{k}_i, \boldsymbol{k}_j; \varkappa)$ depends on the nuclear density and on the momenta $\boldsymbol{k}_i$ and $\boldsymbol{k}_j$ of the interacting pair of nucleons, separated by a distance $r_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. On calling $\boldsymbol{k}_i = \boldsymbol{k}$ and $\boldsymbol{r}_1 = \boldsymbol{r}$, the potential energy of a nucleon of momentum $\boldsymbol{k}$ at a distance $r$ from the center of the nucleus reads

$$\mathcal{V}(k, r; \varkappa) = \frac{2\varkappa^3}{3\pi^2} \int V(\boldsymbol{k}_i, \boldsymbol{k}, \boldsymbol{r}, \boldsymbol{r}_2; \varkappa)\, \mathrm{d}\boldsymbol{r}_2\,, \tag{6.6}$$

*i.e.*

$$\mathcal{V}(k, r; \varkappa) = \frac{4\varkappa^3}{3\pi r}\left\{2r\left(\int_0^{R+r} + \int_0^{R-r}\right) r_{12}^2 V(\boldsymbol{k}_i, \boldsymbol{k}, \boldsymbol{r}, \boldsymbol{r}_2; \varkappa)\, \mathrm{d}r_{12} + \right. \tag{6.7a}$$

$$\left. + \int_{R-r}^{R+r} r_{12}(R^2 - r^2 - r_{12}^2) V(\boldsymbol{k}_i, \boldsymbol{k}, \boldsymbol{r}, \boldsymbol{r}_2; \varkappa)\right\} \mathrm{d}r_{12}\,.$$

for $r \leqslant R$, and

$$\mathcal{V}(k, r; \varkappa) = \frac{4\varkappa^3}{3\pi r} \int_{r-R}^{r+R} (R^2 - r^2 - r_{12}^2 + 2rr_{12}) V(\boldsymbol{k}_i, \boldsymbol{k}, \boldsymbol{r}, \boldsymbol{r}_2) r_{12}\, \mathrm{d}r_{12}\,, \tag{6.7b}$$

for $r \geqslant R$. Using the two-nucleon potential (6.5) into Eqs. (6.7) it is found

$$\mathcal{V}(k, r; \varkappa) = \mathcal{V}^{(1)}(k, \varkappa)\, f(r)\,, \tag{6.8}$$

where $\mathcal{V}^{(1)}(k, \varkappa)$ is given by Eq. (4.7), and

$$f(r) = 1 - (1 + \mu_0 R) \exp[-\mu_0 R] \sinh \mu_0 r/\mu_0 r \qquad r \leqslant R, \tag{6.9a}$$

$$f(r) = (\mu_0 R \cosh \mu_0 R - \sinh \mu_0 R) \exp[-\mu_0 r]/\mu_0 r \qquad r \geqslant R. \tag{6.9b}$$

The nucleus potential energy reads

$$\langle V\rangle = \langle V^{(1)}(\varkappa)\rangle \left\{1 - \frac{3(1+\mu_0 R)(\mu_0 R - \operatorname{tgh}\mu_0 R)}{(\mu_0 R)^3(1+\operatorname{tgh}\mu_0 R)}\right\}. \tag{6.10}$$

At the density minimizing the total energy of the nucleus, Eq. (6.10) simplifies to

$$\langle V\rangle = \langle V^{(1)}(k_F)\rangle[1 - 3/2\mu_0 R]\,, \tag{6.11}$$

and therefore one has

$$B_V = \frac{3}{5}E_F + \frac{\langle V^{(1)}(k_F)\rangle}{A}\,, \tag{6.12a}$$

$$B_S = \left\{-\left(\frac{3}{2\mu_0 r_0}\right)\frac{\langle V^{(1)}(k_F)\rangle}{A}\right\} A^{\frac{1}{3}}\,, \tag{6.12b}$$

where $B_S$ is the average nucleon surface energy. Assuming [12] $B_V = -15.5$ MeV and $B_S = 18A^{-\frac{1}{3}}$ MeV, one has at $r_0 = 1.2$ fermis $\langle V^{(1)}(k_F)\rangle = -35.9A$ MeV and $\mu_0 = 3.5\mu$. In this way it is possible to justify the value $-11.6$ MeV of the minimum of the nucleus total energy, consistent with the $N = 1$ approximation (Table I).

The real part of the nuclear potential, following from the Hamiltonian (6.1), is

$$\mathcal{V}_R(E, r; k_F) = \mathcal{V}_R^{(1)}(E, k_F)F(r)\,, \tag{6.13}$$

where $\mathcal{V}_R^{(1)}(E, k_F)$ is given by Eq. (4.8) and

$$F(r) = f(r)\{\gamma_1(k_F) + [1 - \gamma_1(k_F)]f(r)\}^{-1}. \tag{6.14}$$

The diffuseness of the potential is here ruled by the parameter $\mu_0$. The value $\mu_0 = 3.5\mu$, required to fit the nucleus surface energy, is too large and the radially dependent nucleon effective mass ratio $\gamma_1(r, k_F)$, averaged over all nucleons,

$$\langle\gamma_1(k_F)\rangle = \frac{3}{R^3}\int_0^R \frac{r^2\,\mathrm{d}r}{1 + 2M\omega_1^{(1)}(k_F)f(r)}\,, \tag{6.15}$$

practically coincides with Eq. (5.7). It follows that no substantial modifications to the results listed in Table I are brought about by the radial dependence of the potential, implied by the nucleus Hamiltonian (6.1) with $\mu_0 = 3.5\mu$. In this connection it may be noted that for practical purposes the

radial dependence of $F(r)$, given in Eq. (6.14), is as good as that of the Saxon-Wood potential. It is significant that the range of the effective interaction in the above theory appears to be smaller than that of the interaction between two free nucleons. The two features—small range and no repulsive core—far from being unacceptable, might turn out, on closer inspection, to be just what one should expect of such effective interactions between nucleons in a many nucleon system.

Let us now examine the Schiff-Thirring static, non-linear wave equation for a classical scalar field, with a time independent nucleon source density

$$-\nabla^2\varphi(\boldsymbol{r}) + \mu^2\varphi(\boldsymbol{r}) + \Lambda\varphi^3(\boldsymbol{r}) = g\varrho(\boldsymbol{r})\,, \tag{6.16}$$

derived from the following expression of the total energy associated with the meson field

$$\langle V(\varkappa)\rangle = \int [\tfrac{1}{2}\,(\nabla\varphi)^2 + \tfrac{1}{2}\,\mu^2\varphi^2 + \tfrac{1}{4}\,\Lambda\varphi^4 - g\varrho\varphi]\,\mathrm{d}\boldsymbol{r}\,. \tag{6.17}$$

Neglecting surface effects [$\varphi \equiv \varphi_0$ for $r \lesssim R$], it is possible to solve Eq. (6.17) under the assumption that the condition $\mu^2 > \Lambda\varphi_0^2$ is verified instead of $\mu^2 < \Lambda\varphi_0^2$, as usually assumed ([17,18]). This assumption of ours clearly implies that the field non-linearity alters the two-body forces without making their effects negligible within the nucleus. Thus, it is possible to treat the non-linearity as a perturbation so that the main contribution to the potential energy of a nucleon in the nucleus still arises from the isotopic singlet $\varphi_0 = g\varrho_0\mu^{-2}$. Then, Eq. (6.17) becomes

$$\langle V(\varkappa)\rangle = (\Lambda g^4/4\mu^8)\varrho_0^3 - (g^2/2\mu^2)\varrho_0\,, \tag{6.18}$$

which is nothing but the nucleus potential energy (3.11), evaluated in the $N=3$ approximation under the following conditions

$$C_7^{(3)} = \left(\frac{5}{144\pi^6}\right)\Lambda g^4\mu^{\,8} = \frac{3(E_F - 5B_V)}{64k_F^9}\,, \tag{6.19a}$$

$$C_6^{(3)} = C_5^{(3)} = C_4^{(3)} = C_3^{(3)} = C_2^{(3)} = 0\,, \tag{6.19b}$$

$$C_1^{(3)} = -\left(\frac{2}{3\pi^2}\right)g^2\mu^{\,2} = -\frac{(7E_F - 15B_V)}{5k_F^3}\,. \tag{6.19c}$$

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([18]) B. J. Malenka: *Phys. Rev.*, **86**, 68 (1952); P. Mittelstaedt: *Zeits. f. Phys.*, **137**, 545 (1954).

Eq. (6.18) finally reads

$$\frac{\langle V^{(3)}(\varkappa)\rangle}{A} = \frac{224}{105} C_7^{(3)} \varkappa^9 + \frac{1}{2} C_1^{(3)} \varkappa^3 , \tag{6.20}$$

which evidently underlies the nucleon potential energy (3.6) extended up to $\lambda_{max} = N = 3$. From Eqs. (6.19) one can see that the condition (3.10*a*) is satisfied ($N = 3$, $s = 0$), whereas the condition (3.12) is not: in this regard, both Hamiltonians (6.1) and (6.17) exhibit the same fault. The mathematical consistency of the considered alternative solution of the Schiff-Thirring non-linear theory is proved by the obvious fact that the following inequality is satisfied

$$E_F > -B_V . \tag{6.21}$$

The close connection existing between the Johnson-Teller and the Schiff-Thirring theories as far as the saturation problem is concerned should be apparent by now. In both cases the saturation of the nuclear volume energy and density is insured by the dependence on the limiting momentum $\varkappa$ of the nucleus Hamiltonian brought about by the *ad hoc* introduction of the terms $\mu_0^{-2} f\varphi \sum |\nabla\psi_j|^2$ and $\frac{1}{4}\Lambda\varphi^3$ respectively. In both cases, in addition to the coupling constant $g$, an important role is played by a new parameter, *viz.* by the collective coupling constant $f$ and, in the latter theory, by the non-linearity strenght $\Lambda$. The two parameters, but not only these, come directly from the procedure we have adopted for defining, according to Eqs. (3.8) and (3.9), the solutions of Eq. (1.11). In this connection it is interesting to compare Eqs. (6.3) and (6.19) with Eqs. (3.19): it is seen that in all cases the constant $C_1^{(N)}$ is proportional to the coupling constant $g$, related to two-body forces. Furthermore, in the Johnson-Teller theory it is the mass of the nuclear particles which can formally be considered as dependent on the nuclear density, minimizing the total energy of the nucleus, according to Eqs. (5.7) or (4.9) (nucleon effective mass approximation). Similarly, it is fairly evident that in the Schiff-Thirring theory it is the meson mass which can formally be thought of as dependent on the nuclear density according to the following relation (meson effective mass approximation)

$$\mu^*(\varkappa) = (1 + \Lambda\mu^{-2}\varphi_0^2)^{\frac{1}{2}}\mu , \tag{6.22}$$

which at the density minimizing the total energy of the nucleus explicitly reads

$$\mu^*(k_F) = \left\{\frac{39E_F - 135B_V}{21E_F - 45B_V}\right\}^{\frac{1}{2}}\mu . \tag{6.23}$$

Of course, the value $\mu^*(k_F) = 1.55\mu$, obtained from Eq. (6.23) for $r_0 = 1.2$ fermis and $B_V = -15.5$ MeV, arises from the momentum dependence of the nuclear potential, hidden within the Hamiltonian (6.17). To test the logical consistency of the above considerations it is worth-while pointing out that for $B_V = E_F/5$ the nucleon effective mass $M^*(k_F)$[Eq. (5.7) = Eq. (6.4)], implicit in the Johnson-Teller theory, and the meson effective mass $\mu^*(k_F)$[ Eq. (6.23)], consistent with the Schiff-Thirring theory, reduces to the free nucleon mass $M$ and respectively to the free meson mass $\mu$. The condition $B_V = E_F/5$, which evidently conflicts with nuclear saturation, is implied—as has been shown in Sect. 3 [Eq. (3.14)], by the unrealistic assumption that the solution of Eq. (1.11), representing the potential well in which the nucleons move, be momentum independent. The momentum dependence of the nuclear potential, implicit in the Johnson-Teller and Schiff-Thirring theories, has been evaluated in Table IV in agreement with nuclear saturation.

TABLE IV. – *Momentum dependence of the Johnson-Teller and Schiff-Thirring nuclear potential* (in Mev) *consistent with Eqs.* (2.17) ($B_V = -15.5$ MeV, $r_0 = 1.2$ *fermis*).

| $k \mid k_F$ | 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\mathcal{V}^{(1)}_{J.T.}(k, k_F)$ | — 105.2 | — 101.8 | — 91.3 | — 73.8 | — 49.5 |
| $\mathcal{V}^{(3)}_{S.T.}(k, k_F)$ | — 92.4 | — 91.3 | — 87·8 | — 75.7 | — 49.5 |

Finally, we point out that the contribution to $\mathcal{V}(k, \varkappa)$ due to many-body forces of any order, evaluated in the first approximation of the perturbation method, can always be computed using Eq. (1.7), provided the function $V(r_{12})$ is replaced by an « equivalent » two-body potential $V(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{r}_1, \boldsymbol{r}_2; \varkappa)$, dependent on the nuclear density and on the momenta of the interacting pair of nucleons, separated by a distance $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. We shall not give here the rather complicated mathematical proof of this equivalence and we shall limit ourselves only to point out that, using Eqs. (4.5), it is possible to reproduce the results obtained by DRELL and HUANG (4), who have studied the saturation problem taking into account up to five-body forces. In this case, Eq. (3.11) becomes

$$\frac{\langle V^{(2)}(\eta)\rangle}{A} = \left(\frac{457.4}{\eta^7} + \frac{73.9}{\eta^4} - \frac{845.7}{\eta^5} - \frac{211.2}{\eta^2} + \frac{508.8}{\eta^3}\right) \text{MeV}. \tag{6.24}$$

The minimum of the total energy of the nucleus, $B_V = -12$ MeV, is obtained for $\eta_F = 1.15$, assuming the nucleus kinetic energy (2.18) with $a_C = 0.538/\mu$ [$r_C = 0.38/\mu$]. Eq. (6.24) fullfils the saturation conditions (2.19) [$\langle V^{(2)}(\eta_F)\rangle = -31.4A$ MeV and $\mathcal{V}(\eta_F, \eta_F) = -46.6$ MeV], but is in conflict with

Eqs. (4.5). For instance, the calculated value of the nucleus symmetry energy is $a_3^{(2)}(k_F) = 36.7$ MeV instead of 19.3 MeV. The nucleon potential energy (3.6), from which Eq. (6.24) has been derived, exhibits the characteristic features of the $N = 2$ approximation, being shallower for slow and deeper for fast nucleons in the nucleus, with a maximum at $k = 0.77 k_F$. The momentum dependence of the nuclear potential implied by the Drell and Huang many-body theory of nuclear forces together with that of the potential evaluated from Eqs. (4.16), leading to identical results, is shown in Table V.

TABLE V. – *Comparison between the momentum dependence of the nuclear potential implied by the Drell and Huang many-body theory of nuclear saturation* ($k_F = 1.32\ \mu$, $B_V = -12$ MeV) *and that of the potential computed in the* $N = 2$ *approximation using Eqs.* (4.16), *which, without taking into account the Lenz correction, leads nevertheless to identical results.*

| $k \mid k_F$ | 0 | 0.3 | 0.65 | 0.77 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|
| $\mathcal{V}_{\text{D.H.}}(k, k_F)$ | — 17.6 | — 33.3 | — 69.3 | — 74.0 | — 66.1 | — 46.6 |
| $\mathcal{V}^{(2)}(k, k_F)$ | — 41.4 | — 46.1 | — 53.4 | — 51.5 | — 43.2 | — 30.9 |

The theoretical ambiguity in interpreting the saturation mechanism of the heavy nucleus does not warrant a detailed discussion of the possible consequences emerging from the preceding results, as has been already pointed out at the end of Sect. 4.

## 7. – Concluding remarks.

It is worth-while to emphasize some questions arising from the preceding discussion, to which at present it seems difficult to give a definite answer. First, it may be noted that the expansion of the nuclear potential in even powers of the nucleon momentum implies the dependence, given by Eqs. (3.9), of the nuclear potential coefficients on the limiting momentum $\varkappa$. Thus, the mathematical procedure adopted in Sect. 3 for determining the solutions of Eq. (1.11) apparently reveals a subtle link between the exclusion principle and the invariance prescription of the nuclear potential with respect to time reflection. Second, it is not clear to what extent the Schiff-Thirring and Drell-Huang theories of nuclear saturation can be effectively related to the physical aspects of the nucleon assembly. In fact, the clear traces of the solutions (3.6) and (3.9) of Eq. (1.11), which have been discovered in both of them, do not allow a clear-cut objection against the doubt that these theories play only the mathematical role of providing, on the basis of

preconceived physical ideas, solutions of Eq. (1.11) other, and better, than those following from the first order perturbation calculation with conventional two-body potentials. The question, then, arises as to whether there exists such a two-nucleon potential which leads through Eq. (1.7) to a solution of Eq. (1.11), consistent at least with Eqs. (2.17) and (4.5). This is the key point which emerges from the previous considerations. In conclusion, the promising, but preliminary, results obtained in this paper suggest that more attention should be paid to the nuclear equation (1.11), which stands behind all approaches to the saturation problem so far carried out, from Euler's pioneering work to the recent Brueckner theory.

* * *

The author is pleased to express his sincere appreciation to Prof. P. Budini and to Prof. E. Clementel for stimulating discussions on various topics concerning this paper.

## RIASSUNTO

Il problema della dipendenza dal momento dell'energia potenziale di un nucleone nell'interno del nucleo viene formulato da un nuovo, più generale, punto di vista.

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# Dirac and Gürsey Equations for Bosons.

J. Fiutak

*Department of Physics, Nicholas Copernicus University - Toruń*

(ricevuto il 4 Luglio 1958)

**Summary.** — The aim of this paper is to illustrate a possibility to write over the field equations in a new algebric form. The wave equations for bosons have been obtained in terms of 16-dimensional Clifford algebra $C_4$ (Sect. **2**), and alternatively, in connection with the Gürsey [1] formalism, in terms of the 4-dimensional Clifford algebra $C_2$ (Sect. **3**).

## 1. – Introduction.

It has been shown by Raševsky [2] that Dirac spinors are connected rather with the linear representation of the 16-dimensional Clifford algebra $C_4$, than with the representation of the Lorentz group. It seems that this idea can be generalized. All the field equations may be rewritten in terms of the algebra $C_4$. We shall treat the physical operators, transformations and wave functions of bosons as the elements of the algebra $C_4$. Examples illustrating this idea are presented in Sect. **2**.

There exists also a new possibility to write the wave equations for bosons in terms of the 4-dimensional Clifford algebra $C_2$ according to the Gürsey formalism (Sect. **3**). It is hoped that these equations can be treated as a contribution to modify the usual theory.

(1) F. Gürsey: *Nuovo Cimento*, **7**, 411 (1958); *Nuovo Cimento*, **3**, 988 (1956).
(2) P. K. Raševsky: *Uspehy Mat. Nauk*, **10**, 2 (1955).

## 2. – Dirac equation for bosons.

The most general equation for free boson field with mass $m$ can be wrtiten as:

$$\hat{\partial}\hat{\varphi} = im\hat{\varphi}\,, \tag{2.1}$$

where

$$\hat{\partial} = \gamma_\mu \partial_\mu: \qquad \mu = 0, 1, 2, 3$$

and

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu\,. \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1\,. \tag{2.2}$$

Here $\hat{\varphi}$ is the most general element of $C_4$, and can be written as follows:

$$\hat{\varphi} = \hat{\varphi}_S + \hat{\varphi}_V + \hat{\varphi}_T + \hat{\varphi}_{PV} + \hat{\varphi}_{PS} \tag{2.3}$$

where

$$\hat{\varphi}_S = \varphi \qquad \hat{\varphi}_{PS} = \varphi^P \gamma_5$$

$$\hat{\varphi}_T = \tfrac{1}{2}\varphi_{\mu\nu}\gamma_{\mu\nu}$$

$$\hat{\varphi}_V = \varphi_\mu \gamma_\mu \qquad \hat{\varphi}_{PV} = \varphi^P_\mu \gamma_\mu \gamma_5\,,$$

and

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,, \qquad \gamma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,.$$

Element $\hat{\varphi}$ is called aggregate (according to a proposal of RAŠEVSKY). $C$-numbers (in algebra $C_4$) $\varphi$, $\varphi_\mu$, $\varphi_{\mu\nu}$, $\varphi^P_\mu$, $\varphi^P$ will be called co-ordinates of aggregate $\hat{\varphi}$.

It is well known that the transformations in Lorentz space are represented in algebra $C_4$ by:

$$\hat{\varphi}' = \Lambda\hat{\varphi}\Lambda^{-1}\,, \tag{2.4}$$

where $\Lambda$ is defined for the case of reflection in a plane perpendicular to the direction $a_\mu$ by:

$$\Lambda = \gamma_5\,\hat{a} = \gamma_5 a_\mu \gamma_\mu\,. \tag{2.5}$$

An arbitrary transformation in Lorentz space may be composed of the reflections (2.5). These transformations form a subgroup of all automorphisms of algebra $C_4$.

The equation (2.1) is obviously invariant under the transformation (2.4).

From equation (2.1) one can derive the free field equations for:

I scalar field if $\hat{\varphi} = i\hat{\varphi}_S + \hat{\varphi}_V$

II vector field if $\hat{\varphi} = \hat{\varphi}_V - i\varphi_T$

III pseudo-vector field if $\hat{\varphi} = i\hat{\varphi}_{PV} + \varphi_T$

IV pseudo-scalar field if $\hat{\varphi} = -i\varphi_{PV} + \varphi_P$.

In the cases II and III we obtain also the Lorentz condition.

The equation (2.1) is equivalent to:

$$\hat{\partial}\hat{\varphi} = -m\hat{\pi}, \tag{2.6}$$

$$\hat{\partial}\hat{\pi} = m\hat{\varphi}. \tag{2.7}$$

Here (2.7) may be treated as a field equation, and (2.6) as a definition of the «derivative» $\hat{\pi}$ of a wave-aggregate $\hat{\varphi}$.

We obtain:

I scalar equation if $\hat{\varphi} = \hat{\varphi}_V$ $\hat{\pi} = \hat{\pi}_S = \pi$

II vector equation if $\hat{\varphi} = \hat{\varphi}_V$ $\hat{\pi} = \hat{\pi}_T = -\frac{1}{2}\gamma_{\mu\nu}\pi_{\mu\nu}$

III pseudo-vector equation if $\hat{\varphi} = \hat{\varphi}_{PV}$ $\hat{\pi} = \hat{\pi}_T = -\frac{1}{2}\gamma_{\mu\nu}\pi_{\mu\nu}$

IV pseudo-scalar equation if $\hat{\varphi} = \hat{\varphi}_{PV}$ $\hat{\pi} = \hat{\pi}_S = \gamma_5\pi^P$.

It is possible to describe also the interaction in terms of algebra $C_4$. As an example let us take the Maxwell equation with an external source:

$$\hat{\partial}\hat{A} = \hat{F}, \qquad \hat{\partial}\hat{F} = \frac{1}{c}\hat{j}, \tag{2.10}$$

where

$$\hat{A} = A_\mu\gamma_\mu, \qquad \hat{F} = \tfrac{1}{2}F_{\mu\nu}\gamma_{\mu\nu},$$

$$\hat{j} = j_\mu\gamma_\mu.$$

### 3. – Gürsey's equation for bosons.

GÜRSEY has used the 4-dimensional Clifford algebra $C_2$, to describe the fermion fields.

The most general element of $C_4$ is:

$$A = A_0 - A_i\sigma_i, \qquad A_i\sigma_i = \sum_{i=1}^{3} A_i\sigma_i, \tag{3.1}$$

where $\sigma_i$ are defined by:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1\,, \qquad \sigma_k\sigma_l = i\sigma_m \qquad (k, l, m\text{: cyclic}). \tag{3.2}$$

In this algebra we can introduce the automorphism:

$$A \to \bar{A}^+ = A_0^* + A_i^*\sigma_i \tag{3.3}$$

composed of the anti-automorphisms:

$$A \to \bar{A} = A_0 + A_i\sigma_i\,, \tag{3.4}$$

$$A \to A^+ = A_0^* - A_i^*\sigma_i\,. \tag{3.5}$$

The operations (3.3), (3.4), and (3.5) have, also in a linear representation, been given by GÜRSEY [1].

The automorphism (3.3) plays the role of space inversion in this algebra.

The boson field with finite rest mass $m$ may be described by the system of equations (*):

$$\begin{cases} D\bar{\Phi}^+ = \quad im\Pi\,, \\ D\bar{\Pi}^+ = -\,im\Phi\,, \end{cases} \tag{3.6}$$

where

$$D = \partial_0 - \sigma_i\partial_i\,,$$

$$\Phi = \varphi_0 - \sigma_i\varphi_i\,,$$

$$\Pi = i(\pi_0 - \sigma_i\pi_i)\,.$$

---

(*) One can rewrite (3.6) in the form:

$$\begin{cases} D\bar{\Phi}^+ = -\,m\Pi\,, \\ D\bar{\Pi}^+ = m\Phi\,, \end{cases} \tag{3.6}$$

where

$$\Phi = \varphi_0 - \varphi_i\sigma_i\,, \qquad \Pi = \pi_0 - \pi_i\sigma_i\,.$$

The arbitrarity in form is caused by the invariance of (3.6) against the following transformation:

$$D \to KDK^+\,,$$

$$\Phi \to K\Phi K^+\,,$$

$$\Pi \to K\Pi\bar{K}\,,$$

$$m \to \bar{K}^+ mK^+\,,$$

where $K$ is an arbitrary constant element of $C_2$.

This equation is invariant under the transformation:

$$(3.7)\qquad \begin{cases} A) & \Phi \to L\Phi L^{+}, \\ B) & \Pi \to L\Pi\bar{L}, \end{cases}$$

if

$$\bar{L} = L^{-1}.$$

The transformation (3.7) is, in fact, the Lorentz transformation (see Appendix III).

From (3.6) we obtain scalar or pseudo-scalar equation when

$$(\mathrm{I})\qquad \Phi = \varphi_0 - \varphi_i\sigma_i, \qquad \Pi = i\pi_0,$$

or we obtain vector or pseudo-vector equation (with Lorentz condition) when

$$(\mathrm{II})\qquad \Phi = \varphi_0 - \varphi_i\sigma_i, \qquad \Pi = -i\pi_i\sigma_i.$$

Maxwell equations (together with the Lorentz condition) can be described as follows

$$(3.8)\qquad D\bar{A}^{+} = F, \qquad D\bar{F}^{+} = \frac{1}{c}J,$$

here

$$A = A_0 - A_i\sigma_i,$$
$$J = j_0 - j_i\sigma_i,$$
$$F = (-E_i + iH_i)\sigma_i.$$

In the case when $j_\mu$ is the current of a proton field $p$ we obtain the equations:

$$(3.9)\qquad \begin{cases} D\bar{A}^{+} = F \\ D\bar{F}^{+} = -epp^{+} \end{cases}$$

which are obviously Lorentz-invariant.

* * *

The author acknowledges with pleasure his indebtedness to Professor J. Rayski for his interest in this work.

Thanks are also due to L. Wolniewicz for many discussions.

## Appendix I

From (2.1) we obtain, in the cases I, II, III, IV respectively, the following equations for the co-ordinates of the aggregate

I)
$$\begin{cases} \partial_\mu \varphi = m\varphi_\mu , \\ \partial_\mu \varphi_\mu = -m\varphi , \\ \partial_\mu \varphi_\nu = \partial_\nu \varphi_\mu . \end{cases}$$

II)
$$\begin{cases} \partial_\mu \varphi_\mu = 0 , \\ \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu = m\varphi_{\mu\nu} , \\ \partial_\mu \varphi_{\mu\nu} = -m\varphi_\nu , \\ \partial_\lambda \varphi_{\sigma\varrho} + \partial_\sigma \varphi_{\varrho\lambda} + \partial_\varrho \varphi_{\lambda\sigma} = 0 . \end{cases}$$

III)
$$\begin{cases} \partial_\mu \varphi^P_\mu = 0 , \\ \partial_\mu \varphi^P_\nu - \partial_\nu \varphi^P_\mu = \tfrac{1}{2} m \varepsilon_{\mu\nu}{}^{\sigma\varrho} \varphi_{\sigma\varrho} , \\ \partial_\varrho \varphi_{\lambda\sigma} + \partial_\sigma \varphi_{\varrho\lambda} + \partial_\lambda \varphi_{\sigma\varrho} = m \varepsilon_{\lambda\sigma\varrho}{}^{\mu} \varphi^P_\mu , \\ \partial_\mu \varphi_{\mu\nu} = 0 . \end{cases}$$

IV)
$$\begin{cases} \partial_\mu \varphi^P = m\varphi^P , \\ \partial_\mu \varphi^P_\mu = -m\varphi^P , \\ \partial_\mu \varphi^P_\nu = \partial_\nu \varphi^P_\mu . \end{cases}$$

## Appendix II

We can write the Eq. (3.6) in an explicit form, for scalar and vector field respectively, as follows

I) scalar field

$$\begin{cases} \partial_\mu \varphi_\mu = -m\pi^*_0 , \\ \partial_\mu \pi^*_0 = m\varphi_\mu , \\ \partial_\nu \varphi_\mu = \partial_\mu \varphi_\nu , \end{cases}$$

II) vector field

*a)* $\varphi_\mu$: real; $\pi_k = \pi_{0k} - i\pi_{lm}$; $\pi_{\mu\nu}$: real $(k, l, m$: cyclic)

$$\begin{cases} \partial_\mu \varphi_\mu = 0 , \\ \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu = m\pi_{\mu\nu} , \\ \partial_\mu \pi_{\mu\nu} = - m\varphi_\nu , \\ \partial_\varkappa \pi_{\varrho\lambda} + \partial_\varrho \pi_{\lambda\varkappa} + \partial_\lambda \pi_{\varkappa\varrho} = 0 . \end{cases}$$

*b)* $\varphi_\mu = i\varphi_\mu^P$; $\varphi_\mu^P$: real; $\pi_{\mu\nu}$: real.

$$\begin{cases} \partial_\mu \varphi_\mu^P = 0 , \\ \partial_\mu \varphi_\nu^P - \partial_\nu \varphi_\mu^P = \tfrac{1}{2} m \varepsilon_{\mu\nu}{}^{\sigma\varrho} \pi_{\sigma\varrho} , \\ \partial_\varrho \pi_{\lambda\sigma} + \partial_\lambda \pi_{\sigma\varrho} + \partial_\sigma \pi_{\varrho\lambda} = m \varepsilon_{\lambda\sigma\varrho}{}^{\mu} \varphi_\mu^P , \\ \partial_\mu \pi_{\mu\nu} = 0 . \end{cases}$$

## Appendix III

We shall only examine the transformation properties of tensors (type $B$) because the identity of (3.7) with the Lorentz transformation of a vector (type $A$) is well known.

Let us consider a tensor:

$$\Pi = \pi_0 - \pi_k \sigma_k , \quad \operatorname{Re} \pi_k = \pi_{0k} , \quad \operatorname{Im} \pi_k = - \pi_{lm} . \quad (k, l, m: \text{ cyclic}):$$

It is easy to see that

$$\pi_0' - \pi_k' \sigma_k = \Pi' = L \Pi \bar{L} = \pi_0 - L \pi_k \sigma_k \bar{L} ,$$

so $\pi_0$ is a scalar against (3.7*b*) $B$. On the contrary $\pi_k$ transforms under space rotations like a vector because in this case

$$\bar{L} = L^+ .$$

Considering a Lorentz transformation connected with the motion in the $x^3$-direction with a velocity $v = \operatorname{tgh} \psi$ we obtain

$$L = \cosh \tfrac{1}{2}\psi - \sigma_3 \sinh \tfrac{1}{2} \psi ,$$

which leads to

$$\Pi' = \pi_0 - (\cosh \psi - \sigma_3 \sinh \psi)(\pi_1 \sigma_1 + \pi_2 \sigma_2) - \pi_3 \sigma_3 ,$$

and finally

$$\begin{cases} \pi'_{01} = \cosh\psi\pi_{01} - \sinh\psi\pi_{31}\,, \\ \pi'_{23} = \cosh\psi\pi_{23} - \sinh\psi\pi_{02}\,, \\ \pi'_{02} = \cosh\psi\pi_{02} + \sinh\psi\pi_{23}\,, \\ \pi'_{31} = \cosh\psi\pi_{31} - \sinh\psi\pi_{01}\,, \\ \pi'_{03} = \pi_{03}\,, \\ \pi'_{12} = \pi_{12}\,. \end{cases}$$

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## RIASSUNTO (*)

Scopo del presente lavoro è di illustrare la possibilità di scrivere le equazioni di campo in una nuova forma algebrica. Si sono ottenute le equazioni d'onda pei bosoni in termini dell'algebra $C_4$ di Clifford a 16 dimensioni (Sez. **2**) e, in alternativa, con riferimento al formalismo di Gürsey (1) in termini dall'algebra $C_2$ di Clifford a 4 dimensioni (Sez. **3**).

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(*) *Traduzione a cura della Redazione.*

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# The Diffusion of Argon and Tritium in Liquid Nitrogen.

G. Cini-Castagnoli, G. Pizzella and F. P. Ricci (*)

*Istituto di Fisica dell'Università - Roma*

(ricevuto il 5 Luglio 1958)

**Summary.** — The diffusion coefficient of A and HT in liquid $N_2$ have been measured by using the capillary method, at low concentrations, at constant pressure (about 70 cm Hg), within a range of temperature from 67 to 76 °K. The results can be expressed by the following equations

$$D(\text{A-N}_2) = 730 \exp\left[-\frac{260}{T}\right] \cdot 10^{-6} \text{ cm}^2/\text{s} ,$$

$$D(\text{HT-N}_2) = 1260 \exp\left[-\frac{260}{T}\right] \cdot 10^{-6} \text{ cm}^2/\text{s} .$$

These results are discussed in terms of the dependence of the diffusion coefficient upon the molecular parameters of the different substances.

## 1. – Introduction.

With regard to transport phenomena a great number of experimental data is available concerning viscosity and thermal conductivity of simple liquids; however, very few are available regarding diffusion in the same substances [1-3]. As the understanding of the diffusion mechanism in liquids is essential for any comprehensive theory of transport processes occurring in the liquid state, we have measured the diffusion coefficient of Argon and HT in liquid Nitrogen. These molecules have been chosen because they are among the simplest systems to study from a theoretical point of view.

(*) Now at C.N.R.N.

(1) E. W. Haycock, B. J. Alder and J. H. Hildebrand: *Journ. Chem. Phys.*, **21**, 1601 (1953).

(2) H. Watts, B. J. Alder and J. H. Hildebrand: *Journ. Chem. Phys.*, **23**, 659 (1955).

(3) J. W. Corbett and J. H. Wang: *Journ. Chem. Phys.*, **25**, 422 (1956).

We have investigated the dependence of the diffusion coefficient upon the molecular parameters. The concentration of the tracers has been kept very low in order to avoid concentration effects on the diffusion coefficients.

## 2. – Apparatus and experimental procedure.

In order to obtain good accuracy in the diffusion measurements the capillary method of ANDERSON and SADDINGTON ([4]) has been chosen for this investigation. This method has been adapted for low temperature work. The capillary filled with pure nitrogen, closed at the upper end, is opened at the other end into a bath of nitrogen tagged with argon or HT. The diffusion of the tracer into the capillary is allowed to proceed for a certain length of time, $t$. After diffusion, the liquid is removed from the capillary with a Toepler pump and is analysed in gas form at room temperature.

As will be discussed in the next section, the diffusion coefficient $D$ can be evaluated from the knowledge of the average initial concentration $c_i$ of the tracer in the capillary (which is zero for HT and about 2‰ for A in nitrogen), from the average concentration $c_d$ in the capillary after diffusion, from the concentration $c_s$ of the tracer in the bath, and from the length $L$ of the capillary and the time $t$ of the diffusion.

The apparatus is shown in Fig. 1. The diffusion chamber (of the same kind

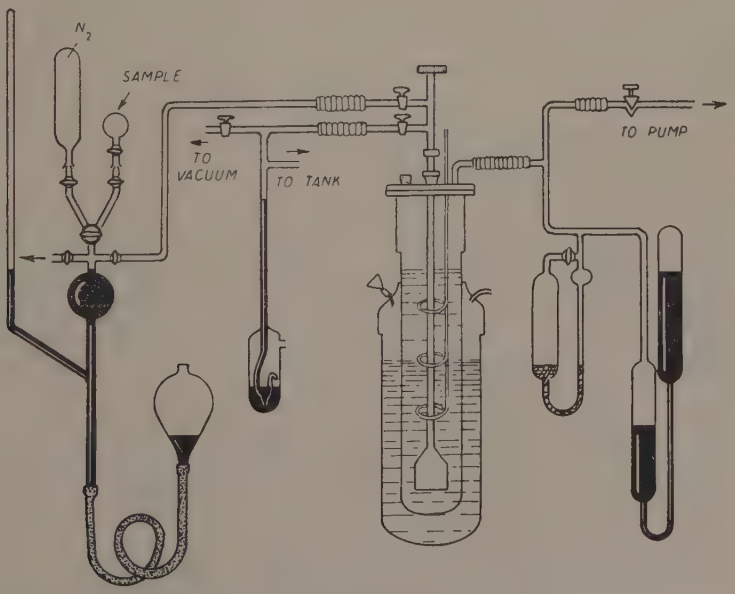


Fig. 1. – Diffusion apparatus.

([4]) J. S. ANDERSON and K. SADDINGTON: *Journ. Chem. Soc. London*, S 381 (1949).

of the one used by other authors (5)) is immersed in a bath of liquid nitrogen. The pressure over the bath could be controlled, thereby providing the temperature regulation for the experiment. This bath is surrounded by another one of liquid air or liquid nitrogen opened to the atmosphere, which reduces the heat inputs to the inner one and makes the temperature control easier. The pressure over the inner bath is measured with a mercury manometer (with an accuracy of 1 mm) and is regulated with a much more sensitive oil differential manometer. The conversion from the nitrogen vapour pressure to temperature is made by LINDER (6) tables (Fig. 2). The temperature is kept constant in each experiment within less than 0.1 °K. Fig. 3 shows schematically the diffusion chamber.

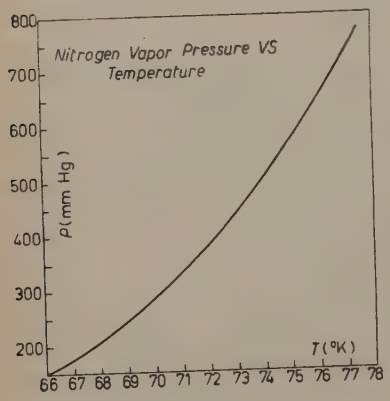


Fig. 2. – Nitrogen vapor pressure *vs.* temperature.

The capillary is a brass tube, 4/10 of mm in diameter, with different lengths in the various runs; it is soldered in a copper block, whose function is to prevent damaging of the capillary when closing it by means of two needle valves.

The copper block ensures a good thermal contact of the capillary with the liquid enclosed in the chamber (whose volume is about 10 $cm^3$).

The tubes are made of german silver to avoid heat flux into the chamber.

The procedure adopted to carry out an experiment was as follows (taking care to maintain constant temperature at all steps): 1) evacuate the system, 2) close the lower end of the capillary, 3) liquify the tagged $N_2$ into the diffusion chamber (*), 4) liquify the pure $N_2$ into the capillary with a Toepler pump and shut the upper valve, 5) open the lower valve of the capillary (at this step the diffusion period begins), 6) close this valve after time $t$ (this ends the diffusion period), 7) remove the pure $N_2$ in the tube leading to the capillary, 8) take out the sample enclosed in the capillary by means of the Toepler pump and send it to a vial.

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(5) BENNAKER, G. CARERI and TACONIS: *Paris Conference on Low Temperatures* (September 1955).

(6) C. T. LINDER: *The Measurement of Low Temperatures*. Res-Report R. 94433-2-A-Westinghouse Res. Lab.-East, Pittsburgh, Penn.

(*) The pressure, the temperature and the concentrations were such that no variation of concentration in the chamber could take place during the time of a diffusion. This was controlled each time removing a sample from the chamber before and after each diffusion.

Analyses were performed in the case of argon-nitrogen diffusion with a Nier type mass spectrometer of the Physics Department of the University of Rome ([7]) and in the case of HT-$N_2$ diffusion with a system based on internal gas G.M. counters ([8]).

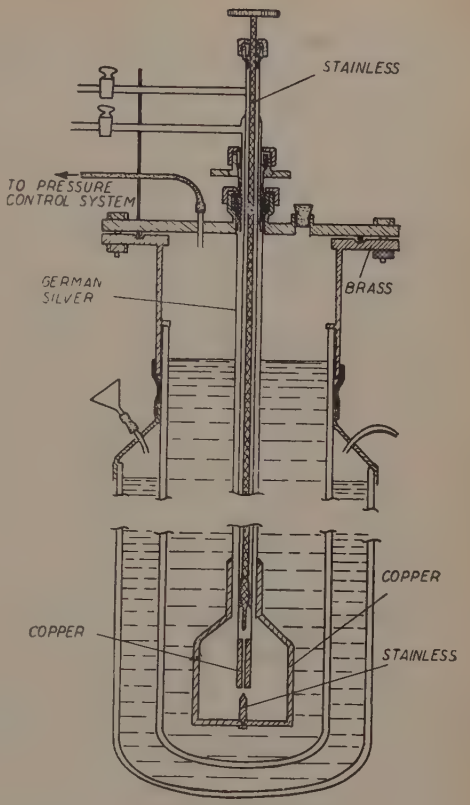


Fig. 3. – Diffusion chamber.

## 3. – Experimental results.

The well known differential equation for the one-dimensional diffusion is

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}, \tag{1}$$

where $c$ is the concentration of tracer at time $t$ and co-ordinate $x$, and $D$ is the diffusion coefficient. With our boundary conditions the solution of eq. (1) is:

$$\frac{c_s - c_{d\,\mathrm{Av}}}{c_s - c_i} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n+1)^2} \exp\left[ \frac{-(2n+1)^2 \pi^2 D t}{4L^2} \right] \right\}, \tag{2}$$

where $c_s$ is the concentration of tracer in the diffusing chamber $c_{d\,\mathrm{Av}}$ is the average concentration in the capillary after time $t$, $c_i$ is the initial concentration of tracer in the capillary at zero time, $L$ is the length of the capillary.

The diffusion coefficient of argon from an about 3% mixture of argon in $N_2$ and of HT from a $10^{-7}$% mixture of HT in $N_2$ was studied over a temperature range of 67 to 76 °K and at a pressure of 70 cm Hg. The pressure was high enough to exclude the formation of bubbles in the diffusion chamber.

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([7]) G. Careri and G. Nencini: *Nuovo Cimento*, **7**, 64 (1950).

([8]) G. Cini-Castagnoli and A. Giardini: to be published in the *Ricerca Scientifica*.

The results of the measurements are listed in Table I for A-$N_2$ and in Table II for HT-$N_2$ and are shown graphically in Fig. 4. The corresponding errors were calculated by taking into account those due to the concentration measurements, and to the length of the capillary.

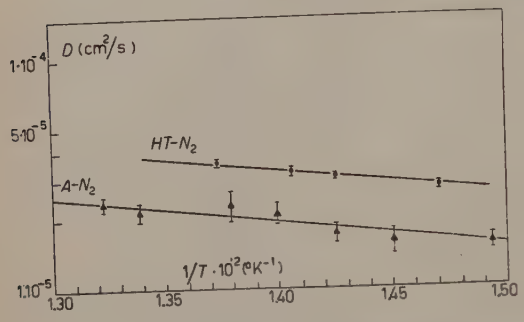


Fig. 4. – Diffusion coefficient of A and HT in liquid $N_2$ as a function of temperature.

TABLE I. – *Experimental results for* A-$N_2$ *diffusion.*

| Run no. | Lenght of the capillary (cm) | Time (s) | Temperature (°K) | D ($cm^2/s$)·$10^5$ |
|---|---|---|---|---|
| 1 | 1.56 ± 0.02 | 7 200 | 74.74 ± 0.15 | 2.38 ± 0.22 |
| 2 | 1.56 ± 0.02 | 10 800 | 71.47 ± 0.07 | 2.10 ± 0.21 |
| 3 | 2.39 ± 0.02 | 7 200 | 68.94 ± 0.10 | 1.59 ± 0.20 |
| 4 | 2.39 ± 0.02 | 4 500 | 72.52 ± 0.05 | 2.26 ± 0.37 |
| 5 | 2.39 ± 0.02 | 7 200 | 66.98 ± 0.10 | 1.57 ± 0.07 |
| 6 | 2.39 ± 0.02 | 10 200 | 70.18 ± 0.08 | 1.75 ± 0.15 |
| 7 | 2.39 ± 0.02 | 5 400 | 75.62 ± 0.05 | 2.41 ± 0.15 |
| 8 | 2.39 ± 0.02 | 7 200 | 74.73 ± 0.05 | 2.0 ± 0.3 |

Fig. 4 shows that $\log D$ is a linear function of $1/T$ at constant pressure. The curves can be expressed by the following equations

$$D(\text{A-N}_2) = (730 \pm 15) \exp\left[-\frac{260 \pm 30}{T}\right] \cdot 10^{-3} \text{ cm}^2/\text{s}\,, \tag{3}$$

$$D(\text{HT-N}_2) = (1260 \pm 13) \exp\left[-\frac{260 \pm 16}{T}\right] \cdot 10^{-3} \text{ cm}^2/\text{s}\,, \tag{4}$$

which were calculated with the least mean square method.

TABLE II. – *Experimental results for* HT-$N_2$ *diffusion.*

| Run no. | Lenght of the capillary (cm) | Time (s) | Temperature (°K) | D $(cm^2/s)\cdot 10^5$ |
|---|---|---|---|---|
| 1 | $1.52 \pm 0.02$ | 4500 | $70.20 \pm 0.07$ | $2.88 \pm 0.30$ |
| 2 | $1.52 \pm 0.02$ | 6600 | $67.40 \pm 0.05$ | $2.81 \pm 0.10$ |
| 3 | $2.37 \pm 0.02$ | 5400 | $71.20 \pm 0.05$ | $3.40 \pm 0.15$ |
| 4 | $2.34 \pm 0.02$ | 4800 | $72.82 \pm 0.06$ | $3.69 \pm 0.10$ |
| 5 | $2.34 \pm 0.02$ | 9000 | $70.08 \pm 0.05$ | $3.16 \pm 0.10$ |

## 4. – Discussion.

4·1. – A, $N_2$ and HT follow accurately (*) the principle of corresponding states in their equilibrium properties [9]; with regard to transport phenomena, the available experimental data are also in satisfactory agreement with this principle [10]. Therefore the only three parameters which characterize the single molecule are its mass $m$ and the two constants $\varepsilon$ and $\sigma$ of the intermolecular potential of the type:

$$\varphi = \varepsilon f\left(\frac{r}{\sigma}\right). \tag{5}$$

Our purpose is to see how the diffusion coefficient can, therefore, be related to the parameters $m$, $\varepsilon$, $\sigma$ (+) characteristic of the different molecules.

4·2. It is well known that the diffusion coefficient in liquids, at constant pressure (see also eq. (3) and (4)), can be written as

$$D = D_0 \exp\frac{\Delta H}{kT}, \tag{6}$$

(*) If one neglects in first approximation quantal effects for HT at liquid $N_2$ temperature.

[9] K. S. PITZER: *Journ. Chem. Phys.*, **7**, 583 (1939); E. A. GUGGENHEIM: *Journ. Chem. Phys.*, **13**, 253 (1945).

[10] G. CINI-CASTAGNOLI, G. PIZZELLA and F. P. RICCI: to be submitted to *Nuovo Cimento.*

(+) The values of $\varepsilon$ and $\sigma$ are given by J. DE BOER: *Physica*, **14**, 139 (1948). For substances not listed there, the parameters have been calculated from the critical points.

where $D_0$ and $\Delta H$ (the activation energy of the process) are two constants, $k$ is the Boltzmann constant and $T$ the absolute temperature.

At low concentrations and at a given pressure, for perfect liquids (9) we have

$$\Delta H = F(\varepsilon_1, \sigma_1, m_1; \varepsilon_{i1}, \sigma_{i1}, m_i; k) , \tag{7}$$

$$D_0 = G(\varepsilon_1, \sigma_1, m_1; \varepsilon_{i1}, \sigma_{i1}, m_i; k) , \tag{8}$$

where

$$\varepsilon_{i1} = \sqrt{\varepsilon_1 \varepsilon_i} , \tag{9}$$

$$\sigma_{i1} = \frac{\sigma_1 + \sigma_i}{2} , \tag{10}$$

as often suggested (*). We have indicated with subscripts $i$ and 1 the molecular parameters of the tracer and those of the abundant liquid respectively. Under the stated conditions the form of the functions $F$ and $G$ does not depend upon the nature of the two liquids.

4·3. – Our experimental data (cfr. eq. (3) and (4)) show that the activation energy does not depend upon the tracer. The height of the barrier to be overcome is, therefore, at least for low concentrations, equal for HT and A molecules. This result is in agreement with previous ones reported by other authors for liquids at higher temperatures (2,11). It follows that eq. (7) can be written as

$$\Delta H = F(\varepsilon_1, \sigma_1, m_1; k) . \tag{11}$$

Furthermore since the only combination of $\varepsilon_1$, $\sigma_1$, $m_1$ and $k$ with the dimension of an energy is just $\varepsilon_1$, it follows necessarily

$$F(\varepsilon_1, \sigma_1, m_1; k) = C\varepsilon_1 , \tag{12}$$

where $C$ is a constant which does not depend upon the nature of the liquid.

Taking, for $N_2$, $\varepsilon/K = 96.6$ °K, from our values of the activation energy we find:

$$\Delta H = (2.6 \pm 0.1)\varepsilon_1 . \tag{13}$$

---

(*) $\varepsilon_{i1}$ and $\sigma_{i1}$ are the constants of the intermolecular potential between unlike molecules which can be written in the form (5).

(11) A. PAOLETTI and M. VICENTINI: *Diffusion of Lead, Tin, and Indium in Liquid Indium* (private communication).

This equation (*) is fairly well satisfied for $CCl_4$ [2] at least with the same degree of accuracy, with which $CCl_4$ conforms to the principle of corresponding states in its other properties.

4·4. – In what concerns eq. (8) our results, which are in agreement with those of other experiments, show a clear dependence of $D_0$ upon the tracer; for this reason any model involving many molecules motion for describing the diffusion mechanism in liquids must be excluded.

WATTS *et al*, [2] have proposed an interpretation of their results by stating that « since the activation energies are the same, the relative magnitude of the diffusion of $I_2$ and $CCl_4$ may be referred to the relative frequency with which the barrier is overcome. This depends in large part both upon their relative velocities, given by the square root of the inverse ratio of their masses and upon the inverse ratio of their cross-sections, taken as the 2/3 power of their liquid molar volumes ».

Although they do not give any explicit formula this statement seems to be equivalent to writing:

$$D_0 = g(\varepsilon_1, \sigma_1, m_1; k)\frac{1}{\sigma_i^2}\sqrt{\frac{1}{m_i}}. \tag{14}$$

Since our data give

$$\frac{D_0(\text{A-N}_2)}{D_0(\text{HT-N}_2)} = 0.58 \pm 0.02\,,$$

while eq. (14) gives a value 0.23 we conclude that the relation (14) is inconsistent with our experiment. A similar attempt to collect the available data can be done by the empirical relation:

$$D_0 = g'(\varepsilon_1, \sigma_1, m_1; k)\frac{1}{\sigma_{i1}}\sqrt{\frac{\varepsilon_{i1}}{m_{i1}}}, \tag{15}$$

where $\varepsilon_{i1}$, $\sigma_{i1}$ are given in eq. (9), (10) and $m_{i1} = (2m_1m_i)/(m_1+m_i)$, (see Table III).

TABLE III.

| | exp. value | calc. by eq. (15) |
|---|---|---|
| $\frac{D_0(\text{A-N}_2)}{D_0(\text{HT-N}_2)}$ | $0.58 \pm 0.02$ | 0.58 |
| $\frac{D_0(\text{CCl}_4\text{-CCl}_4)}{D_0(\text{I}_2\text{-CCl}_4)}$ | 0.96 | 0.96 |

(*) In this equation $C$ seems not to depend on pressure, as we can deduce from the results of ref. [1].

4·5. – It may be interesting to point out some consequences deriving from eq. (15). As $D_0$ is expressed in cm²/s, $g'(\varepsilon_1, \sigma_1, m_1; k)$ must have the dimensions of the square of a length. This is only possible if $g'$ does not depend upon $\varepsilon_1$, $m_1$, $k$. In this case eq. (15) becomes:

$$D_0 = P\sigma_1^2 \sqrt{\frac{\varepsilon_{i1}}{m_{i1}}} \frac{1}{\sigma_{i1}} \tag{16}$$

where $P$ is a constant without dimensions which does not depend upon the type of the liquid. From our data we find $P = 9 \cdot 10^3$, if $\sigma$ is measured in Å, $\varepsilon$ in erg and $m$ in (a.m.u.).

Eq. (6) with $\Delta H$ and $D_0$ calculated from eq. (13) and (16) becomes:

$$D = 9 \cdot 10^3 \frac{\sigma_1^2}{\sigma_{i1}} \sqrt{\frac{\varepsilon_{i1}}{m_{i1}}} \exp\left[-\frac{2.6\varepsilon_1}{kT}\right] \text{cm}^2/\text{s} . \tag{17}$$

In the particular case of self diffusion the eq. (17) holds for $i = 1$.

For the self diffusion of argon we obtain at 84.3 °K and at $\sim 1$ atm:

$$D = (1.6 \pm 0.3) \cdot 10^{-5} \text{ cm}^2/\text{s} .$$

The experimental value ([3]) is $D = (2.07 \pm 0.06) \cdot 10^{-5}$ cm²/s. The error on the calculated value of $D$ is due to the uncertainties with which the constants $C$ and $P$ are given from our experimental data. The agreement is as satisfactory, as one might expect, considering that $N_2$ deviates slightly from the principle of corresponding states in its transport properties ([10]).

4·6. – Finally, it is of interest, with reference to general theories of transport processes in liquids, to examine the quantity $D\eta/T$ as a function of temperature ($\eta$ is the viscosity). From the present results and from the values of $\eta$ for $N_2$ ([12]), we can see that $D\eta/T$ is constant within 5% for A-$N_2$ and for HT-$N_2$. If one calculates $r$ from the Stokes relation $D\eta/kT = 1/6\pi r$, one finds $sr_A = 1.3$ Å, $r_{HT} = 0.8$ Å. The ratio $r_A/r_{HT}$ is very different from the ratio $\sigma_A/\sigma_{HT}$.

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([12]) N. S. RUDENKO and L. W. SCHUBNIKOW: *Sov. Phys.*, **6**, 670 (1934).

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We are deeply indebted to Prof. G. Careri for having suggested the present investigation and some useful advices concerning the experimental technique. Thanks are due to F. Duprè and A. Giardini for their help in performing some measurements.

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## RIASSUNTO (*)

Si sono misurati a basse concentrazioni e a pressione costante (circa 70 cm Hg) in un campo di temperature da 67 a 76 °K i coefficienti di diffusione dell'A e di HT in $N_2$ liquido servendosi del metodo del capillare. I risultati si possono esprimere con le seguenti equazioni

$$D(\text{A-N}_2) = 730 \exp\left[-\frac{260}{T}\right] \cdot 10^{-6} \text{ cm}^2/\text{s} ,$$

$$D(\text{HT-N}_2) = 1260 \exp\left[-\frac{260}{T}\right] \cdot 10^{-6} \text{ cm}^2/\text{s} .$$

Si discutono tali risultati in termini della dipendenza del coefficiente di diffusione dai parametri molecolari delle differenti soluzioni.

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(*) *Traduzione a cura della Redazione.*

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# On the Causal Propagation Function of a Dirac Field.

E. Ferrari and G. Jona-Lasinio

*Istituto di Fisica dell'Università - Roma*
*Istituto Nazionale di Fisica Nucleare - Sezione di Roma*

(ricevuto il 10 Luglio 1958)

**Summary.** — It is shown that the causal propagation function of a Dirac field can be split in two functions which satisfy a system of two integral equations similar to the couple of Low equations for the symmetrical charged fixed-source meson theory, in the one-meson approximation. These equations are solved and discussed.

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1. – It has been shown by K. W. Ford [1] that the Fourier transform of the causal propagation function of a boson field, interacting relativistically in a given, but unspecified way with other fields, satisfies an integral equation which is formally identical with the Low equation for a neutral fixed-source meson theory, in the one-meson approximation. The equation contains in place of the known cut-off function, an unknown function, fixed in principle by the interaction. Ford shows that a condition for the existence of the solution of this equation leads to the limitation on the asymptotic behaviour of the « vertex function » previously found by other methods by Lehmann, Symanzik and Zimmermann [2].

In this paper we shall attempt to show that an analogous situation occurs also with the causal propagation function of a Dirac field, in the sense that, by means of a convenient decomposition of it suggested by relativistic invariance requirements, two functions can be defined, which satisfy a system of two integral equations, which is formally similar to the couple of Low equations

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[1] K. W. Ford: *Phys. Rev.*, **105**, 320 (1957).
[2] H. Lehmann, K. Symanzik and W. Zimmermann: *Nuovo Cimento*, **2**, 425 (1955).

for the symmetrical charged fixed-source meson theory, in the one-meson approximation. In this case the integral equations contain two unknown functions which have the role of a natural cut-off built in in the theory. Also in this case the integrability conditions of this system lead, if the two unknown functions are approximated by limiting the intermediate states, and assuming a pseudoscalar interaction of the Fermi field with a pseudoscalar neutral boson field, to a condition for the asymptotic behaviour of the « vertex function » for final fermion and boson on the energy shell (3).

2. – The starting point of our discussion is a treatment of the $S^{(+)'}$ function

$$-iS^{(+)'}_{\alpha\beta}(x-x') = \langle \Omega \Psi_\alpha(x) \overline{\Psi}_\beta(x') \Omega \rangle$$

(where $\Psi_\alpha$, $\overline{\Psi}_\beta$ are the field operators and the brackets indicate an average value in the vacuum state $\Omega$) analogous to the one followed in ref. (2) in the case of the $\Delta^{(+)'}$ function. We put:

$$\langle \Omega \Psi_\alpha(x) \overline{\Psi}_\beta(x') \Omega \rangle = \sum_\nu \langle \Omega, \Psi_\alpha(x) \Phi^{(\nu)}_{in} \rangle \langle \Phi^{(\nu)}_{in}, \overline{\Psi}_\beta(x') \Omega \rangle$$

introducing into the sum a complete set of physical intermediate states, belonging to the same value of baryonic quantum number. The label $\nu$ indicates the number and the type of the particles present in the state.

It is easily seen that this expression can be written in the form

)1) $$\langle \Omega \Psi_\alpha(x) \overline{\Psi}_\beta(x') \Omega \rangle = \left[ \int S'_F(x-\xi) F(\xi-\eta) \bar{S}'_F(\eta - x') \, \mathrm{d}^4\xi \, \mathrm{d}^4\eta \right]_{\alpha\beta},$$

where $S'_F$, $\bar{S}'_F$ are the causal propagation function defined by

$$S'_{F\alpha\beta}(x-y) = \langle \Omega, T(\Psi_\alpha(x) \overline{\Psi}_\beta(y)) \Omega \rangle \qquad \bar{S}'_{F\alpha\beta}(x-y) = \langle \Omega, \bar{T} (\Psi_\alpha(x) \overline{\Psi}_\beta(y)) \Omega \rangle .$$

$T$ and $\bar{T}$ indicate invariant chronologically and antichronologically ordered products respectively.

The function $F$ can be thought of as a sum of the contributions of the intermediate states with a given number $n$ of present particles. It is still a $4 \times 4$ matrix.

We now define

(2) $$F(x) = \sum_{n \geqslant 2} F_n(x) .$$

(3) In ref. (2) this limitation has been stated without proof.

As we write $F(x)$ in this form, we drop from the right-hand side of (1) the first term ($n=1$) which reproduces the free function $-iS^{(+)}(x)$. Of course this function must be subtracted from the left-hand side too.

Now introducing the Fourier transforms we have

$$-i[S^{(+)\prime}(k) - S^{(+)}(k)] = S'_F(k)F(k)\bar{S}'_F(k)\,. \tag{3}$$

We see that $F(k)$ contains only positive frequencies. Recalling also relativistic invariance, we can put ($\boldsymbol{k} = \sum_\mu \gamma^\mu k_\mu$)

$$F(k) = \theta(k_0)\{(i\boldsymbol{k} + \sqrt{-k^2})F_1(-k^2) + (i\boldsymbol{k} - \sqrt{-k^2})F_2(-k^2)\}, \tag{4}$$

$$S'_F(k) = \tfrac{1}{2}\{(i\boldsymbol{k} + \sqrt{-k^2})S_1(-k^2) + (i\boldsymbol{k} - \sqrt{-k^2})S_2(-k^2)\}. \tag{5}$$

It turns out to be

$$\bar{S}'_F(k) = \tfrac{1}{2}\{(i\boldsymbol{k} + \sqrt{-k^2})S_1^*(-k^2) + (i\boldsymbol{k} - \sqrt{-k^2})S_2^*(-k^2)\} \tag{5'}$$

and from (3) and (4), (5), (5') we get the equation

$$2\pi\{(i\boldsymbol{k} - \sqrt{-k^2})\sigma_1(-k^2) + \varrho_2(-k^2)\} = i\boldsymbol{k}(-k^2)\{F_1|S_1|^2 + F_2|S_2|^2\} + \\ + \sqrt{-k^2}(-k^2)\{F_1|S_1|^2 - F_2|S_2|^2\},$$

where $\sigma_1$ and $\varrho_2$ are the mass spectral functions introduced first by LEHMANN [4].

When we separate the part dependent on the $\gamma$-matrices in the last equation, we get two independent equations

$$\begin{cases} 2\pi\sigma_1(-k^2) = (-k^2)\{F_1|S_1|^2 + F_2|S_2|^2\} \\ 2\pi[\varrho_2(-k^2) - \sqrt{-k^2}\sigma_1(-k^2)] = \sqrt{-k^2}(-k^2)\{F_1|S_1|^2 - F_2|S_2|^2\} \end{cases} \tag{6}$$

through which it can be stated directly that $F_1(-k^2)$ and $F_2(-k^2)$ are real and positive-definite functions [5]. In the following we shall use the notation $\sqrt{-k^2} = \chi$. This variable must be considered always positive, since it stands for a mass. Now we recall the spectral representation of the function $S'_F(k)$ [3,4]

$$S'_F(k) = -i\frac{i\boldsymbol{k} - M}{-\chi^2 + M^2} - i\int_{(2M)^2}^{\infty} \frac{(i\boldsymbol{k} - \varkappa)\sigma_1(\varkappa^2) + \varrho_2(\varkappa^2)}{-\chi^2 + \varkappa^2 - i\varepsilon}\,\mathrm{d}\varkappa^2\,. \tag{7}$$

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[4] H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

[5] We recall the limitations for $\sigma_1$ and $\varrho_2$

$$\varrho_2(-k^2) \geqslant 0\,, \qquad 2\sqrt{-k^2}\sigma_1(-k^2) - \varrho_2(-k^2) \geqslant 0\,.$$

Substituting the right-hand sides of (6) for $\sigma_1$ and $\varrho_2$ we get the following integral equations

$$(8)\qquad \begin{cases} \chi S_1(\chi^2) = \dfrac{i}{\chi+M} - \dfrac{i}{\pi}\displaystyle\int_{2M}^{\infty} \frac{F_1(\varkappa^2)\,|\,S_1(\varkappa^2)\,|^2\varkappa^3}{\varkappa-\chi-i\varepsilon}\,\mathrm{d}\varkappa + \frac{i}{\pi}\int_{2M}^{\infty} \frac{F_2(\varkappa^2)\,|\,S_2(\varkappa^2)\,|^2\varkappa^3}{\varkappa+\chi}\,\mathrm{d}\varkappa\,, \\ \chi S_2(\chi^2) = \dfrac{i}{\chi-M} - \dfrac{i}{\pi}\displaystyle\int_{2M}^{\infty} \frac{F_2(\varkappa^2)\,|\,S_2(\varkappa^2)\,|^2\varkappa^3}{\varkappa-\chi-i\varepsilon}\,\mathrm{d}\varkappa + \frac{i}{\pi}\int_{2M}^{\infty} \frac{F_1(\varkappa^2)\,|\,S_1(\varkappa^2)\,|^2\varkappa^3}{\varkappa+\chi}\,\mathrm{d}\varkappa\,. \end{cases}$$

The functions $F_1(\varkappa^2)$ and $F_2(\varkappa^2)$ are unknown but not arbitrary. With the positions

$$\chi=\omega\,,\quad \varkappa=\omega'\,,\quad M=\omega_0\,,\quad \chi F_i(\chi^2)=G_i(\omega)\,,\quad -i\chi S_i(\chi^2)=\lambda_i(\omega) \qquad (i=1,2)$$

we find

$$(9)\qquad \begin{cases} \lambda_1(\omega) = \dfrac{1}{\omega+\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\,\lambda_1(\omega')\,|^2}{\omega'-\omega-i\varepsilon}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\,\lambda_2(\omega')\,|^2}{\omega'+\omega}\,\mathrm{d}\omega'\,, \\ \lambda_2(\omega) = \dfrac{1}{\omega-\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\,\lambda_2(\omega')\,|^2}{\omega'-\omega-i\varepsilon}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\,\lambda_1(\omega')\,|^2}{\omega'+\omega}\,\mathrm{d}\omega'\,. \end{cases}$$

These equations are formally identical with the Low equations for the symmetric charged meson theory, with only one « bound state term » (with energy $\omega_0 \neq 0$), except for the fact that in our case there are two « spectral functions » $G_1$ and $G_2$. The integration of the system can be performed using the method introduced by Castillejo, Dalitz and Dyson (6). The analytical procedure is shown in the Appendix. We get

$$(10)\qquad \begin{cases} \dfrac{1}{\lambda_1(\omega)} = \dfrac{1}{C} + \dfrac{\omega}{\pi}\displaystyle\int_{2\omega_0}^{\infty}\left[\frac{G_1(\omega')}{\omega'(\omega'-\omega-i\varepsilon)} + \frac{G_2(\omega')}{\omega'(\omega'+\omega)}\right]\mathrm{d}\omega' - R(-\omega)\,, \\ \dfrac{1}{\lambda_2(\omega)} = -\dfrac{1}{C} + \dfrac{\omega}{\pi}\displaystyle\int_{2\omega_0}^{\infty}\left[\frac{G_2(\omega')}{\omega'(\omega'-\omega-i\varepsilon)} + \frac{G_1(\omega')}{\omega'(\omega'+\omega)}\right]\mathrm{d}\omega' + R(\quad\omega)\,. \end{cases}$$

(6) L. Castillejo, R. H. Dalitz and F. J. Dyson: *Phys. Rev.*, **101**, 453 (1956).

where

$$R(\omega) = \omega\left[A + \sum_i \frac{R_i^{(1)}}{\omega_i^{(1)}(\omega_i^{(1)} - \omega)} + \sum_i \frac{R_i^{(2)}}{\omega_i^{(2)}(\omega_i^{(2)} + \omega)}\right],$$

$C$, $A$, $R_i^{(j)}$, $\omega_i^{(j)}$ are real constants (see Appendix).

From integrability conditions, we find that $G_1(\omega)$ and $G_2(\omega)$ must satisfy the following limitations

$$\int^{\infty} \frac{G_1(\omega)}{\omega^2}\,\mathrm{d}\omega < \infty \qquad \int^{\infty} \frac{G_2(\omega)}{\omega^2}\,\mathrm{d}\omega < \infty\,. \tag{11}$$

In the Appendix is also given a proof that (10) is a solution of system (9) under the conditions (11).

We think that the above formula (10) may be of some interest, because it expresses the two functions appearing in the causal propagator of a fermion field in terms of other two functions, about which, although they are still unknown, something more can be said. At this point we have both $\Delta_F'$ (v. ref. (1)) and $S_F'$ expressed in terms of some « spectral » functions. By some convenient limitation on the number of intermediate states, some approximate methods of calculation might be developed.

3. – The limitations (11) imply a convergence condition on $F_1(\chi^2)$ and $F_2(\chi^2)$ as $\chi \to \infty$. We can find the same convergence conditions following another procedure, namely the one used in ref. (2). If in system (7) we substitute $S_F'(k)$ as a function of $\sigma_1(\chi^2)$ and $\varrho_2(\chi^2)$ (instead of $\sigma_1$ and $\varrho_2$ as a function of $S_1$ and $S_2$) we get a system of two integral equations, which show some analogy with the equation (8) of ref. (2). Even though it does not seem possible to get an explicit solution for $\sigma_1$ and $\varrho_2$ separately, a convergence condition on $F_1$ and $F_2$, analogous to (11), can be obtained.

If we start from (11) and consider the intermediate physical states with $n = 2$ (containing a nucleon and a pseudoscalar meson) we get a condition on the asymptotic behaviour of the « vertex function », defined as in ref. (2)

$$\langle \Omega T(\Psi(x)A(y)\overline{\Psi}(z))\,\Omega\rangle = \int S_F'(x-\xi)\Delta_F'(y-\eta)\Gamma_5(\xi,\eta,\zeta)S_F'(\zeta - z)\,\mathrm{d}^4\xi\,\mathrm{d}^4\eta\,\mathrm{d}^4\zeta\,.$$

In particular if one makes the following decomposition of the Fourier transform of the vertex function (in the case in which the meson is on the energy shell)

$$\Gamma_5(p,k) = \gamma^5\left[\left\{(\boldsymbol{p}-\boldsymbol{k}) + \sqrt{-(p-k)^2}\right\} f_1(-(p-k)^2) + \right.$$
$$\left. + \left\{(\boldsymbol{p}-\boldsymbol{k}) - \sqrt{-(p-k)^2}\right\} f_2(-(p-k)^2)\right]$$

it is found that $|q|f_1(-q^2)$ and $|q|f_2(-q^2)$ must vanish as $-q^2 \to \infty$. $q = p - k$ is the momentum of the nucleon not on the energy shell.

* * *

We thank Prof. M. CINI for having suggested the problem and for his constant helpful advice.

## APPENDIX

We start from the equations

$$(9)\qquad \begin{cases} \lambda_1(\omega) = \dfrac{1}{\omega+\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\lambda_1(\omega')|^2}{\omega'-\omega-i\varepsilon}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\lambda_2(\omega')|^2}{\omega'+\omega}\,\mathrm{d}\omega' , \\ \lambda_2(\omega) = \dfrac{1}{\omega-\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\lambda_2(\omega')|^2}{\omega'-\omega-i\varepsilon}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\lambda_1(\omega')|^2}{\omega'+\omega}\,\mathrm{d}\omega' , \end{cases}$$

We define the functions $h_1(\omega)$ and $h_2(\omega)$ of the complex variable $\omega$

$$\begin{cases} h_1(\omega) = \dfrac{1}{\omega+\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\lambda_1(\omega')|^2}{\omega'-\omega}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\lambda_2(\omega')|^2}{\omega'+\omega}\,\mathrm{d}\omega' , \\ h_2(\omega) = \dfrac{1}{\omega-\omega_0} - \dfrac{1}{\pi}\displaystyle\int_{2\omega_0}^{\infty} \frac{G_2(\omega')\,|\lambda_2(\omega')|^2}{\omega'-\omega}\,\mathrm{d}\omega' + \frac{1}{\pi}\int_{2\omega_0}^{\infty} \frac{G_1(\omega')\,|\lambda_1(\omega')|^2}{\omega'+\omega}\,\mathrm{d}\omega' , \end{cases}$$

which are meromorphic in the complex $\omega$-plane cut along the real axis from $2\omega_0$ to $+\infty$ and from $-2\omega_0$ to $-\infty$.

These functions satisfy a set of conditions identical or analogous to the ones listed in the paper by CASTILLEJO, DALITZ and DYSON [6] under *a*) *b*) *c*). We shall emphasize only the property which is of particular interest in our case: for complex values of $\omega$ ve have

$$(12)\qquad \operatorname{Im} h_{1,2}(\omega) = k_{1,2}(\omega)\operatorname{Im}\omega ,$$

where $k_{1,2}(\omega) < 0$ because of $G_{1,2} > 0$.

If now we define

$$H_{1\,2}(\omega) = \frac{1}{h_{1\,2}(\omega)} ,$$

we see that the functions $H_{1,2}(\omega)$ satisfy a set of conditions analogous to the ones listed in ref. ( ) under $(A)$ to $(E)$. In particular it can be stated from (12) that $H_{1,2}(\omega)$ are generalized $R$-functions.

It follows directly that we can use the « theorem $H$ » quoted in ref. ([6]) in order to write down the expression of $H_{1\,2}(\omega)$ explicitly in the form ([7]).

$$(13) \qquad H_{1\,2}(\omega) = -(-1)^{1\,2}C^{-1} + \frac{\omega}{\pi}\int_{2\omega_0}^{\infty}\left[\frac{G_{1\,2}(\omega')}{\omega'(\omega'-\omega)} + \frac{G_{2\,1}(\omega')}{\omega'(\omega'+\omega)}\right]\mathrm{d}\omega' + \\ + (-1)^{1\,2}R[(-1)^{1\,2}\omega]\,,$$

provided the following integrals are convergent

$$(14) \qquad \int^{\infty}\frac{G_1(\omega)}{\omega^2}\,\mathrm{d}\omega \qquad \int^{\infty}\frac{G_2(\omega)}{\omega^2}\,\mathrm{d}\omega \qquad \int_0\frac{G_2(\omega)-G_1(\omega)}{\omega}\,\mathrm{d}\omega\,.$$

The function $R(\omega)$ is defined through

$$R(\omega) = A\omega + \sum_i \frac{R_i^{(1)}\omega}{\omega_i^{(1)}(\omega_i^{(1)}-\omega)} + \sum_i \frac{R_i^{(2)}\omega}{\omega_i^{(2)}(\omega_i^{(2)}+\omega)} = A\omega + \sum_n \frac{R_n\omega}{\omega_n(\omega_n-\omega)}\,,$$

$A$, $R_i^{(1,2)}\omega_i^{(1,2)}$ are positive real constants. The latter represent the points where $h_{1,2}(\omega)$ vanish, and can be finite or infinite in number, or even not present at all. They must satisfy some conditions, in order to let $H_{1\,2}(\omega)$ have the required analytical properties. We do not report them here: they can be deduced following the line shown in ref. ([6]). As far as their number and values are concerned, nothing can be said: it does not seem justifiable to suppose that the physical arguments that can be applied to the case of the Low equations ( ) can be extended to ours.

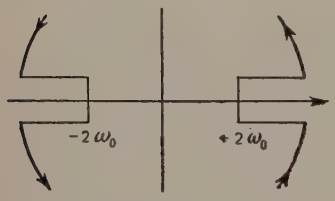


Fig. 1.

Formula (10) of the text follows directly from (13) when one puts $\omega \to \omega + i\varepsilon$ and recalls that as $\omega$ tends to real values $> 2\omega_0$ from the upper half plane, $h_{1,2}(\omega)$ tends to the function $\lambda_{1\,2}(\omega)$. The first two equations (14) are the convergence conditions (11). In order to conclude, we can obtain a proof of the fact that (10) is a solution of system (9) under the conditions (11), if we make the contour integrals

$$\frac{1}{2\pi i}\int_C \frac{\mathrm{d}z'}{(z'-z)H_{1,2}(z')}\,,$$

where $C$ is the integration path in the complex $z$-plane shown in Fig. 1.

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([7]) $C$ is defined by $(-1)^{1,2}C = h_{1,2}(0)$. In the case $C=0$, the first term of the solution is to be substituted with $C'/\omega$, where $C'$ is a real constant $\leqslant 0$.

([8]) R. Haag: *Nuovo Cimento*, **5**, 203 (1957); F. J. Dyson: *Phys. Rev.*, **106**, 157 (1957); D. B. Fairlie and J. C. Polkinghorne: *Nuovo Cimento*, **8**, 345 (1958).

In evaluating the result, we must recall the following « crossing property » of the functions $H_{1,2}(\omega)$

$$H_1(-\omega) = -H_2(\omega)\,,$$

which can be deduced directly from their definition.

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## RIASSUNTO

Si dimostra che la funzione causale di propagazione di un campo fermionico può essere espressa mediante due funzioni che soddisfano un sistema di due equazioni integrali, simile alla coppia di equazioni di Low per la teoria mesonica di sorgente fissa per mesoni carichi, nell'approssimazione di un solo mesone. Queste equazioni sono risolte e discusse.

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# General Relativity: Relative Standard Mass, Momentum, Energy and Gravitational Field in a General System of Reference.

C. Cattaneo

*Istituto Matematico dell'Università - Pisa*

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**Summary.** — In this paper we show that the equations of the motion of a free test particle in General Relativity, with respect to any physical system of reference, can always have the classical interpretation:

*Time derivative of momentum = mass × gravitational field.*

The result is reached by means of suitable definitions of the different dynamical quantities relative to the chosen system of reference and by means of the systematic use of a differential operation that generalizes the usual absolute differentiation. In the gravitational field there are present three parts: one deriving from a scalar potential, one similar to Coriolis' apparent field, and one part deriving from a vector potential.

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The aim of this work is to show that the relativistic equations of motion of a free particle can be given a form and an interpretation similar to that of classic dynamics, without any limiting hypothesis either regarding the system of reference or the motion of the particle or the intensity of the gravitational field.

The result is based mainly on two points: 1) the adoption of a *standard time* relative to the chosen fluid system of reference and of other standard relative quantities (mass, velocity, momentum) defined in accordance; 2) the systematic use of a particular differential operation: *partial transverse derivation*, normal or covariant, with respect to co-ordinate lines $x^4 = \text{var}$.

With this adoption we shall demonstrate that the equations of motion

relative to a system of reference however chosen may always be put in the form

*Time derivative of momentum = mass × gravitational field.*

In the gravitational field we shall distinguish a part deriving from a scalar potential, a part originating from a vector potential if this is non-stationary, and, lastly, a part altogether similar to Coriolis' apparent field which is present when the time tracks of the chosen system of reference do not form a normal congruence.

It is well known that results of this type have already been found to be valid, *in the first order of approximation*, in conditions of slow motion and weak fields: they will be established here in an exact manner and without any limitations.

## 1. – General considerations on the space-time continuum and on the physical systems of reference.

1·1. *Physical system of reference.* – Let $V_4$ be the space-time continuum, $x^1$, $x^2$, $x^3$, $x^4$ a physically admissible system of co-ordinates, with $x^4 = ct$ [1] the time co-ordinate and $x^\alpha$ ($\alpha = 1, 2, 3$) the space co-ordinates and

$$ds^2 = g_{ik}\,dx^i\,dx^k \tag{1}$$

(the sum is extended to the value of indexes from 1 to 4) [2] the fundamental quadratic form, that we shall suppose hyperbolic of the type $+++-$. The physical admissibility of the co-ordinates can be expressed by the conditions:

$$g_{44} < 0\,, \qquad g_{\alpha\beta}\,dx^\alpha\,dx^\beta > 0 \tag{2}$$

(for every choice of the $dx^\alpha$ not simultaneously nil; $\alpha$ and $\beta$ vary from 1 to 3), and permits the co-ordinate lines $x^4 = \text{var.}$ to be interpreted as time tracks of $\infty^3$ ideal particles which in their totality constitute the *physical system of reference* $S$ associated to the chosen system of co-ordinates. The particles of $S$—*reference particle*—considered in the same instant $t = x^4/c$ form a hypersurface of $V_4$, of the spatial kind, which is generally called the physical space relative to that instant.

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[1] The letter $c$ represents the speed of light in a vacuum infinitely far from matter.

[2] In the whole of this paper we shall follow the convention that the latin indexes shall vary from 1 to 4, whereas greek indexes shall vary from 1 ot 3.

However, it should be noted that the meaning of physical entity $S$ of reference—*fluid of reference*—is a quality, more than of the hypersurface $x^4 = \text{const}$, of the totality of the reference particles considered during their whole existence.

For each particle of $S$ the time variable $x^4$ (or $t = x^4/c$) can be regarded as being furnished by a clock connected to the particle and working in an arbitrary manner with the only condition that along each time track $x^4 = \text{var.}$, $dt/d\tau$ shall be positive, $d\tau$ being a proper time interval for the particle examined.

A clock so described is called co-ordinate clock and the time $t$ it gives is called *co-ordinate time*. Naturally, all the clocks are understood to be linked to each other in some fashion in order to respect, for the co-ordinates $x^\iota$ and $x^4$ the quality of admissibility.

From now on we shall consider only physically admissible co-ordinates.

A general change of co-ordinates in $V_4$

$$x^{i'} = x^{i'}(x^1, x^2, x^3, x^4) \qquad (i' = 1, 2, 3, 4) \tag{3}$$

does not always imply a corresponding change of the « fluid of reference ». In order that the latter shall remain unchanged it is necessary and sufficient that the change shall leave unaltered the co-ordinate lines $x^4 = \text{var.}$, and that it therefore be of the type:

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, x^3), \quad x^{4'} = x^{4'}(x^1, x^2, x^3, x^4) \qquad (\alpha' = 1, 2, 3) \tag{4}$$

that will be called a change of co-ordinates *internal* to the given physical system of reference. The change (4) can obviously be decomposed into a change of the sole space co-ordinates

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, x^3), \qquad x^{4'} = x^4 \tag{5}$$

by a change of the type

$$x^{\alpha'} = x^\alpha, \quad x^{4'} = x^{4'}(x^1, x^2, x^3, x^4) \tag{6}$$

which, instead, corresponds to a change in the co-ordinate time, *i.e.* a change of the co-ordinate clock which is linked to each particle.

1·2. *Natural decomposition of a 4-vector in an assigned physical system of reference. Standard metrics, dynamic potential.* – Having assigned a physically admissible system of co-ordinates, in a point of $V_4$ let $\boldsymbol{\gamma}$ be the unit vector tangent to the line $x^4 = \text{var.}$ and oriented towards the future; its contravariant and covariant components, having regard to the conditions $g_{ik}\gamma^i\gamma^k = -1$,

are respectively

$$\gamma^i = \left(0, 0, 0, \frac{1}{\sqrt{-g_{44}}}\right) \qquad (i = 1, 2, 3, 4), \tag{7}$$

and

$$\gamma_i = g_{ik}\gamma^k = \frac{g_{i4}}{\sqrt{-g_{44}}} \qquad (i = 1, 2, 3, 4). \tag{8}$$

Though this is unnecessary, it can be at once formally verified that the $\gamma^i$'s and $\gamma_i$'s behave, in the presence of an internal transformation of coordinates, like the contravariant and, respectively, the covariant components of a vector. Naturally, this no longer occurs in the presence of a general transformation of the type (3), since such a transformation is as a rule accompanied by a transformation of the vector $\boldsymbol{\gamma}$ itself.

Having assigned, in any point of $V_4$, a 4-vector $\boldsymbol{V}$, this may be resolved unically, as is well known [3], into the sum of a vector $\boldsymbol{A}$ parallel to $\boldsymbol{\gamma}$ and a vector $\boldsymbol{N}$ normal to it. We have:

$$\boldsymbol{A} = -(\boldsymbol{V}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\,, \qquad \boldsymbol{N} = \boldsymbol{V} - \boldsymbol{A} \tag{9}$$

or also, in terms of components,

$$A^k = -(V^i\gamma_i)\gamma^k\,, \qquad N^k = V^k + (V^i\gamma_i)\gamma^k\,, \tag{10}$$

$$A_k = -(V^i\gamma_i)\gamma_k\,, \qquad N_k = V_k + (V^i\gamma_i)\gamma_k\,. \tag{11}$$

It is well known that the space-time norms of $\boldsymbol{A}$ and $\boldsymbol{N}$ are called respectively the *time norm* and the *space norm* of the 4-vector $\boldsymbol{V}$. The results of calculation are:

$$\text{Time norm of } \boldsymbol{V} = g_{ik}A^iA^k = -(V^i\gamma_i)^2 < 0\,, \tag{12}$$

$$\text{Space norm of } \boldsymbol{V} = g_{ik}N^iN^k = \gamma_{ik}V^iV^k = \gamma_{\alpha\beta}V^\alpha V^\beta > 0\,, \tag{13}$$

where

$$\gamma_{ik} = g_{ik} + \gamma_i\gamma_k\,. \tag{14}$$

For the components of the symmetric tensor of the second order $\gamma_{ik}$ with which, instead of with $g_{ik}$, the space norm of a 4-vector is calculated, a similar remark is valid to the one made above regarding the vector $\boldsymbol{\gamma}$: they are transformed like the components of a tensor in the presence of an internal trans-

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[3] Cfr. A. LICHNEROWICZ: *Théories relativistes de la gravitation et de l'électromagnétisme* (Paris, 1955), p. 8.

formation of co-ordinates; and not, of course, with respect to a general transformation that usually implies a transformation of the vector $\boldsymbol{\gamma}$.

It should be noted that, with regard to (14), we have identically:

$$\gamma_{\alpha 4} = \gamma_{4\alpha} = \gamma_{44} = 0 \,, \tag{15}$$

which permits us (see (13)) to employ, for calculating the space norm of $V$, its sole spatial contravariant components. As can be immediately verified, the spatial components $\gamma_\cdot$ of the vector $\boldsymbol{\gamma}$ and the components $\gamma_{\cdot\beta}$ of the tensor $\gamma_{ik}$ behave respectively like the components of a 3-vector and of a spatial tensor (having 9 components) in a purely spatial internal transformation of the type (5). In such a transformation $g_{44}$, and with it $\gamma_4$, remains unchanged.

The tensor $\gamma_{ik}$ (of which the sole components $\gamma_{\cdot\beta}$ are not null) can be called *standard spatial metric tensor* relative to the assigned physical system of reference. Similarly, its associated quadratic form

$$\mathrm{d}\sigma^2 = \gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \,, \tag{16}$$

that gives the space norm of the general vector $(\mathrm{d}x^i)$, will be called *standard spatial metric* of the system of reference $S$. The unit 4-vector $\boldsymbol{\gamma}$ will be given the denomination, which we shall justify later, of *time metric vector*, associated to the chosen system of reference. It may be given, for other reasons, the name of *dynamical four-potential* associated to the physical system of reference $S$. More particularly, $\gamma_4$ is called the *scalar potential*, leaving the name of vector potential to the 3-vector $\gamma_\alpha$, or better, as we shall see, to the 3-vector $\gamma_\alpha/\gamma_4$.

It should be noted that the element that characterizes a physical system of reference $S$ is in fact the field of the unit 4-vectors $\boldsymbol{\gamma}$ that defines at the same time the time tracks of the reference particles and, in every point, the standard metrics, both of time and space, that are associated to the system of reference.

1·3. *Conditions for the elimination, in an assigned physical system of reference, of the 3-potential vector* $\gamma_\alpha$. – The components $\gamma_i$ of the 4-vector $\boldsymbol{\gamma}$, having regard to the unit character of this last, can never be simultaneously null. Some time, however, under certain conditions for the system of reference $S$, it is possible to obtain, through a suitable internal transformation of co-ordinate time, the simultaneous annulment of the $\gamma_\alpha$'s: it is possible, that is, to obtain a time-orthogonal co-ordinate system. This possibility is, as we know, equivalent to the condition that the time tracks of the reference par-

ticles, $x^4 = \text{var.}$, shall form a *normal congruence* and this condition can be expressed by the following equations (4)

$$\Omega_{\alpha\beta} = 0 \tag{17}$$

which are equivalent to the annulment of the skew-symmetric spatial tensor of the second order whose components are (5):

$$\Omega_{\alpha\beta} = \left(\partial_\alpha - \frac{\gamma_\alpha}{\gamma_4}\partial_4\right)\frac{\gamma_\beta}{\gamma_4} - \left(\partial_\beta - \frac{\gamma_\beta}{\gamma_4}\partial_4\right)\frac{\gamma_\alpha}{\gamma_4}, \tag{18}$$

proper to the reference system adopted. The condition, in the form (17) (18), is owed to J. VON WEYSSENHOFF (6).

## 2. – The laws of motion of a free particle and their particular interpretations with respect to a physical system of reference.

**2·1.** *The laws of the motion of a free particle* (7). – Let us consider in $V_4$ a test particle $M$, which, by definition, is subjected to the influence of the gravitational field through having itself no influence on the field.

We shall call $m_0$ its proper mass, that is to say, the proper mass it would have in an empty region and very far from matter; $d\tau$ a proper interval of time ($d\tau^2 = -ds^2/c^2$) $\boldsymbol{U} - U^i \equiv dx^i/d\tau$ ($i = 1, 2, 3, 4$) the 4-velocity of the particle, $\boldsymbol{P} \equiv P^i \equiv m_0 U^i$ its 4-momentum.

This stated, the law of the motion of $M$ can be summarized in the vectorial equation:

$$\frac{D\boldsymbol{P}}{d\tau} = 0, \tag{20}$$

equivalent to one or the other of the two differential systems in the contravariant or covariant components of $P$:

$$\frac{DP^i}{d\tau} = 0, \tag{21}$$

$$\frac{DP_i}{d\tau} = 0, \tag{22}$$

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(4) See, for example, C. MØLLER: *The Theory of Relativity* (Oxford, 1952), p. 248 foll.

(5) The symbol $\partial_i$ stands for $\partial/\partial x^i$.

(6) J. v. WEYSSENHOFF: *Bull. Acad. Polon.*, Sec. A, 252 (1937).

(7) Regarding this Sect. **2·1**, see MØLLER: *op. cit.*, p. 288 segg.

where $D$ is the symbol of absolute differentiation in $V_4$. More explicitly (21) may be written:

$$\frac{DP^i}{d\tau} = \frac{dP^i}{d\tau} + \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} P^h U^k = 0 , \tag{23}$$

$$\frac{DP_i}{d\tau} = \frac{dP_i}{d\tau} - [ik, h] P^h U^k = \frac{dP_i}{d\tau} - \frac{1}{2} \partial_i g_{hk} P^h U^k = 0 , \tag{24}$$

where $[ik, h]$ and $\left\{ \begin{matrix} i \\ hk \end{matrix} \right\}$ are Christoffel's symbols of the first and second kind.

The absolute interpretation of (20) and its equivalent equations in terms of components is immediate: *the absolute path of $M$ is a time-like geodesic of $V_4$.*

In order to give equations (23) and (24) an interpretation relative to the system of reference the following definitions are usually introduced

Relative 3-vector velocity:

$$u^\alpha = \frac{dx^\alpha}{dt} . \tag{25}$$

Square of its magnitude:

$$u^2 = \gamma_{\alpha\beta} u^\alpha u^\beta \tag{26}$$

(the adoption of the spatial metric $\gamma_{\alpha\beta}$ in place of $g_{\alpha\beta}$ should be noted).

Relative mass:

$$m' = m_0 \Gamma , \tag{27}$$

$\Gamma$ being the ratio of the elementary interval of co-ordinate time $dt$ to the corresponding proper time interval $d\tau$:

$$\Gamma = \frac{dt}{d\tau} = \left[ \left\{ -\gamma_4 - \frac{\gamma_\alpha u^\alpha}{c} \right\}^2 - \frac{u^2}{c^2} \right]^{-\frac{1}{2}} . \tag{28}$$

Relative 3-momentum:

$$p^\alpha = P^\alpha = m' u^\alpha . \tag{29}$$

Its covariant components:

$$p_\alpha = \gamma_{\alpha\beta} p^\beta ; \tag{30}$$

Total relative energy:

$$H = -cP_4 = m'c^2\gamma_4\left(\gamma_4 + \frac{\gamma_\alpha u^\alpha}{c}\right). \tag{31}$$

Scalar gravitational potential:

$$-\chi = \frac{(1+g_{44})c^2}{2}. \tag{32}$$

3-potential vector:

$$\gamma_\alpha = \frac{g_{\alpha 4}}{\sqrt{-g_{44}}}. \tag{33}$$

With regard to these definitions it should be noted that the relative mass $m'$ is seen to be dependent, through the factor $\Gamma$, not only on the motion of the particle, but also on the gravitational field.

It should be further noted that whilst the contravariant components $p^\alpha$ of the relative 3-momentum are defined as identical to the corresponding contravariant components of the 4-vector $\boldsymbol{P}$, its covariant components $p_\alpha$ are seen to be not identical, as a rule, to the corresponding covariant components $P_\alpha$ of $\boldsymbol{P}$.

It will be found useful to recall that, in the case of weak fields and slow motion, the energy $H$ may be expressed approximately:

$$H \cong m_0c^2 + \tfrac{1}{2}m_0u^2 + m_0\chi\,, \tag{34}$$

where there appear three terms which may obviously be interpreted as representing rest energy, kinetic energy and potential energy.

The adoption of the metric tensor $\gamma_{\alpha\beta}$ in place of $g_{\alpha\beta}$ for the measurement of spatial vectors suggests the introduction, for a general field of spatial vectors $q_\alpha$ (or $q^\alpha$), also of a covariant spatial derivation $\delta_\mu q_\alpha$ and an absolute spatial differentiation $\mathrm{d}^* q_\alpha$ which may be respectively defined in the following manner:

$$\delta_\mu q_\alpha = \partial_\mu q_\alpha - \gamma_{\alpha\mu,\lambda}q^\lambda\,, \tag{35}$$

$$\mathrm{d}^* q_\alpha = \mathrm{d}q_\alpha - \gamma_{\alpha\mu,\lambda}q^\lambda\,\mathrm{d}x^\mu\,, \tag{36}$$

where $\gamma_{\alpha\mu,\lambda}$ are the three-dimensional Christoffel symbols of the first kind made up with $\gamma_{\alpha\beta}$'s.

If the spatial vector is given in contravariant form $q^\alpha$ we shall have, instead:

$$\delta_\mu q^\alpha = \partial_\mu q^\alpha + \gamma^\alpha{}_{\mu\lambda}q^\lambda\,, \tag{37}$$

$$\mathrm{d}^* q^\alpha = \mathrm{d}q^\alpha + \gamma^\alpha{}_{\mu\lambda}q^\lambda\,\mathrm{d}x^\mu\,. \tag{38}$$

Here $\gamma^\alpha{}_{\mu\lambda}$ are the analogous symbols of the second kind.

With these definitions we realize, for example, that in the case of a co-ordinate time-orthogonal system ($\gamma_\alpha = 0$) the first three equations (24) take on the form [8]

$$\frac{\mathrm{d}^* p_\alpha}{\mathrm{d}t} = - m' \partial_\alpha \chi \,, \tag{39}$$

which is perfectly similar to that of newtonian mechanics.

Another important instance in which the first three equations (24) have a significant relative interpretation is that in which, although the spatial vector $\gamma_\alpha$ is different from zero, the test particle is supposed *instantaneously* at rest in the chosen system of reference ($u^\backprime = 0$): at this instant, in fact, the equations (24) can be put in the form:

$$\frac{\mathrm{d}^* p_\alpha}{\mathrm{d}t} = - m' \partial_\alpha \chi + m' c^2 \gamma_4 \partial_4 \gamma_\alpha \,, \tag{40}$$

where the right side still has the expressive shape of a real gravitational force, proportional to the mass (which is now reduced to the mass of relative rest) and formally similar to the ponderomotive force which an electro-magnetic field exerts on a charged particle in relative rest (which justifies the names of scalar and potential vector given respectively to $\chi$ and $\gamma_\alpha$).

If we give up both the hypothesis of time-orthogonality of the co-ordinate system of reference and the hypothesis that the particle be instantaneously at relative rest, equations (24) can still have an expressive shape *of first approximation* in the simultaneous hypothesis of *slow motion* and *weak fields*. In such conditions, disregarding terms of the 2-nd order in the quantities $u^\backprime/c$, $\gamma_\alpha$, $\chi$, equations (24) may be given the form:

$$\frac{\mathrm{d}^* p_\alpha}{\mathrm{d}t} = - m' (\partial_\alpha \chi + c^2 \partial_4 \gamma_\alpha - c \omega_{\alpha\beta} u^\beta) \,, \tag{41}$$

where

$$\omega_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha \,. \tag{42}$$

In (41) the two corresponding terms that appear in (40) have, added to them, a term, $c m' \omega_{\alpha\beta} u^\beta$, that is of the Coriolis' force type.

Nevertheless (41), though expressive, is only an approximate equation, which is no longer valid when we study a general motion in a general field.

In the following sections we shall show how a more suitable definition of relative quantities and the systematic use of a certain differential operator

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[8] Cfr. MØLLER: *op. cit.*, p. 291.

enables us to give the equations of motion a physically expressive form even in the most general conditions of motion and with regard to the most general physical system of reference.

## 3. – Relative interpretation of the laws of the motion of a free particle with regard to a general fluid of reference.

3·1. *Transverse, ordinary and covariant, derivation and differentiation.* – We shall now make a digression to introduce a differential operation that will be useful later on.

Given in $V_4$ any scalar function $f(x^1, x^2, x^3, x^4)$, let us consider its gradient, having covariant components $\partial_i f$. Let $(\mathrm{d}x^i)$ be, in a general point of $V_4$, any 4-vector subject to the sole condition of being perpendicular to $\boldsymbol{\gamma}$ (unit 4-vector tangent the co-ordinate line $x^4 = \text{var.}$); this may be expressed by

$$\gamma^i \mathrm{d}x_i \equiv \gamma_i \, \mathrm{d}x^i = 0 \tag{43}$$

and implies, having regard to (7),

$$\mathrm{d}x^4 = -\frac{\gamma_\alpha \mathrm{d}x^\alpha}{\gamma_4}, \qquad \mathrm{d}x_4 = 0 . \tag{44}$$

The differential of $f$ with respect to the vector $(\mathrm{d}x^i)$,

$$\mathrm{d}f = \partial_i f \, \mathrm{d}x^i \tag{45}$$

having regard to (44), may also be written:

$$\mathrm{d}f = \left( \partial_\alpha f - \frac{\gamma_\alpha}{\gamma_4} \partial_4 f \right) \mathrm{d}x^\alpha . \tag{46}$$

We are thus naturally induced to consider the operator

$$\tilde{\partial}_i = \partial_i - \frac{\gamma_i}{\gamma_4} \partial_4 . \tag{47}$$

Through this operator the differential of $f$ with respect to a vector $(\mathrm{d}x^i)$ orthogonal to $\boldsymbol{\gamma}$ may be expressed by

$$\mathrm{d}f = \tilde{\partial}_i f \, \mathrm{d}x^i = \tilde{\partial}_\alpha f \, \mathrm{d}x^\alpha \tag{48}$$

which employs the sole spatial components of $(\mathrm{d}x^i)$.

The quantities $\tilde{\partial}_i f$, the fourth one of which is always nil, are the covariant components of a 4-vector which, for every $f$, is perpendicular to $\boldsymbol{\gamma}$: we shall therefore call it the *transverse gradient* (with respect to the direction of $\boldsymbol{\gamma}$) of the function $f$. It can also be easily seen that it is nothing else than the projection vector of the 4-vector $(\partial_i f)$ on the tangent 3-space orthogonal to $\boldsymbol{\gamma}$. The operation $\tilde{\partial}_\alpha$ defined by (47), will therefore be called *transverse partial derivation*.

The differential operation we have now introduced can be extended, in a natural way, to the case where, instead of a scalar function, we consider a 4-vectorial field $(v_k)$ which be orthogonal, in every point, to the vector $\boldsymbol{\gamma}$, a condition which implies:

$$v^4 = -\frac{\gamma_\alpha v^\alpha}{\gamma_4}, \qquad v_4 = 0\,. \tag{49}$$

Having called $\nabla_i v_k$ the covariant derivative of $(v_k)$ and considered, as we did a short while ago, a vector $(\mathrm{d}x^i)$ orthogonal to $\boldsymbol{\gamma}$, let us apply to the vector $Dv_k = \nabla_i v_k \,\mathrm{d}x^i$—the absolute differential of $(v_k)$—the decomposition (9) of n. 2. Having stated, as before:

$$\nabla_i v_k \,\mathrm{d}x^i = A_k + N_k \tag{50}$$

we find

$$A_k = \frac{\gamma_k}{\gamma_4}\,(\nabla_i v_4 \mathrm{d}x^i)\,; \tag{51}$$

$$N_k = \left(\nabla_i v_k - \frac{\gamma_k}{\gamma_4}\,\nabla_i v_4\right)\mathrm{d}x^i = Dv_k - \frac{\gamma_k}{\gamma_4}\,Dv_4\,, \tag{52}$$

where we have not yet taken into account (44) and (49).

The spatial vector $N_k$ (in respect of which we always have $N_4 = 0$) we shall call absolute transverse differential (with respect to the vector $\boldsymbol{\gamma}$) of the vector $(v_k)$.

If we limit, as is now natural, the variation of $k$ to the sole indexes 1 to 3, and take into account (44) and (49), (52) may, through simple calculations, be written explicitly:

$$N_\beta = \tilde{\nabla}^*_\alpha v_\beta \,\mathrm{d}x^\alpha\,, \tag{53}$$

where we have put

$$\tilde{\nabla}^*_\alpha v_\beta = \tilde{\partial}_\alpha v_\beta - \left[\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\} - \frac{\gamma_\beta}{\gamma_4}\left\{\begin{matrix}\lambda\\ \alpha 4\end{matrix}\right\} - \frac{\gamma_\alpha}{\gamma_4}\left\{\begin{matrix}\lambda\\ 4\beta\end{matrix}\right\} + \frac{\gamma_\alpha\gamma_\beta}{\gamma_4^2}\left\{\begin{matrix}\lambda\\ 44\end{matrix}\right\}\right] v_\lambda\,. \tag{54}$$

The $\tilde{\nabla}^*_\nu v^\beta$ represent, in consequence of the arbitrarity of the $\mathrm{d}x^\nu$'s in (53), a spatial tensor (belonging, that is, to the three dimensional tangent subspace orthogonal to $\boldsymbol{\gamma}$) which we shall call *transverse covariant derivative* of the vector $(v_k)$.

A simple but rather long calculation would show that $\tilde{\nabla}^*_\nu v_\beta$ can be given the following expression

$$\tilde{\nabla}^*_\alpha v_\beta = \tilde{\partial}_\alpha v_\beta - \left\{ \begin{matrix} \tilde{\lambda} \\ \alpha\beta \end{matrix} \right\}^* v_\lambda , \tag{55}$$

similar to that of an usual covariant derivative, where the Christoffel's symbols are formed by means of the metric tensor $\gamma_{\nu\beta}$ and the partial derivatives $\partial_\nu$ are systematically substituted by $\tilde{\partial}_\alpha$:

$$\left\{ \begin{matrix} \tilde{\lambda} \\ \alpha\beta \end{matrix} \right\}^* = \gamma^{\lambda\varrho} \widetilde{[\alpha\beta, \varrho]}^* , \tag{56}$$

$$\widetilde{[\alpha\beta, \varrho]}^* = \tfrac{1}{2}(\tilde{\partial}_\alpha \gamma_{\beta\varrho} + \tilde{\partial}_\beta \gamma_{\varrho\alpha} - \tilde{\partial}_\varrho \gamma_{\alpha\beta}) . \tag{57}$$

Renouncing here to demonstrate (55), let us consider only the case in which the vectors $\mathrm{d}\boldsymbol{P} = (\mathrm{d}x^i)$ and $\boldsymbol{v} = (v^i)$ are parallel to each other, as well as, of course, normal to $\boldsymbol{\gamma}$

$$\mathrm{d}x^i = c v^i . \tag{58}$$

In this case the equation (52) furnishes directly for the transverse differential of $\boldsymbol{v}$ the expression

$$N_\beta = \partial_i v_\beta \mathrm{d}x^i - \left\{ \begin{matrix} h \\ i\beta \end{matrix} \right\} v_h \mathrm{d}x^i - \frac{\gamma_\beta}{\gamma_4} \partial_i v_4 \, \mathrm{d}x^i + \frac{\gamma_\beta}{\gamma_4} \left\{ \begin{matrix} h \\ i4 \end{matrix} \right\} v_h \mathrm{d}x^i = \mathrm{d}v_\beta - \tag{59}$$

$$- \frac{1}{2} \tilde{\partial}_\beta g_{\lambda\mu} v^\lambda \mathrm{d}x^\mu + \frac{1}{2} \frac{\gamma_\mu}{\gamma_4} \tilde{\partial}_\beta g_{\lambda 4} v^\lambda \mathrm{d}x^\mu + \frac{1}{2} \frac{\gamma_\lambda}{\gamma_4} \tilde{\partial}_\beta g_{4\mu} v^\lambda \mathrm{d}x^\mu - \frac{1}{2} \frac{\gamma_\lambda \gamma_\mu}{\gamma_4^2} \tilde{\partial}_\beta g_{44} v^\lambda \mathrm{d}x^\mu .$$

Remembering now that we have

$$\gamma_\lambda = \frac{g_{\lambda 4}}{\sqrt{-g_{44}}} , \qquad \gamma_4 = -\sqrt{-g_{44}} \tag{60}$$

we find

$$\tilde{\partial}_\beta \gamma_\lambda = - \frac{\tilde{\partial}_\beta g_{\lambda 4}}{\gamma_4} + \frac{1}{2} \frac{\gamma_\lambda \tilde{\partial}_\beta g_{44}}{\gamma_4^2} , \tag{61}$$

and

$$\frac{1}{2} \frac{\gamma_\mu}{\gamma_4} \tilde{\partial}_\beta g_{\lambda 4} v^\lambda \mathrm{d}x^\mu + \frac{1}{2} \frac{\gamma_\mu}{\gamma_4} \tilde{\partial}_\beta g_{4\lambda} v^\mu \mathrm{d}x^\lambda - \frac{1}{2} \frac{\gamma_\lambda \gamma_\mu}{\gamma_4^2} \tilde{\partial}_\beta g_{44} v^\lambda \mathrm{d}x^\mu = - \frac{1}{2} \tilde{\partial}_\beta (\gamma_\lambda \gamma_\mu) v^\lambda \mathrm{d}x^\mu . \tag{62}$$

Having regard to (14), it follows,

$$N_\beta = \tilde{\mathrm{d}} v_\beta - \tfrac{1}{2}\,\tilde{\partial}_\beta \gamma_{\lambda\mu} v^\lambda \,\mathrm{d}x^\mu \,. \tag{63}$$

On the right hand side of (63) there appears an expression which is formally correspondent to that of the absolute differential of the spatial 3-vector $v_\beta$ with respect to the 3-vector $\mathrm{d}x^\nu = c v^\nu$. It is, however, calculated employing as a metric tensor what we have called the standard metric space tensor $\gamma_{\alpha\beta}$, and substituting the ordinary partial derivatives $\partial_\nu$ with the $\tilde{\partial}_\nu$.

This absolute differentiation we shall denote with the symbol $\tilde{d}^*$,

$$\tilde{\mathrm{d}}^* v_\beta = \tilde{\mathrm{d}} v_\beta - \tfrac{1}{2}\,\tilde{\partial}_\beta \gamma_{\lambda\mu} v^\lambda \,\mathrm{d}x^\mu \,, \tag{64}$$

where the two different marks $\sim$ and $*$ remind us that we are using $\partial_\alpha$'s and $\gamma_{\alpha\beta}$. From what has been said so far the name of transverse absolute differential of the vector $v_\beta$ with respect to the vector $\mathrm{d}x^\nu$, parallel to it, is seen to be justified. It should not be forgotten that the formula (64) is essentially subordinate to the hypothesis that the two spatial vectors $v^\nu$ and $\mathrm{d}x^\nu$ shall be parallel to each other.

3·2. *Standard relative quantities.* – In the definitions of the preceding Sect. 2·1 there is systematically involved as relative time interval between two infinitely close events $x^i$ and $x^i + \mathrm{d}x^i$, the quantity $\mathrm{d}t = \mathrm{d}x^4/c$, *i.e.*, the co-ordinate time interval, which however has no real physical significance.

Leaving $\mathrm{d}t$ its name of co-ordinate time interval, we shall call *standard time, relative to a given physical system of reference S*, the quantity

$$\mathrm{d}T = -\frac{1}{c}\gamma_i \mathrm{d}x^i \,, \tag{65}$$

or, in an equivalent manner,

$$\mathrm{d}T = -\left(\gamma_4 \mathrm{d}t + \frac{1}{c}\gamma_\beta v^\beta\right). \tag{66}$$

The definition is justified by the following circumstances:

1) Its square, multiplied by $-c^2$, is nothing else than the time norm of the 4-vector $(\mathrm{d}x^i)$, with respect to the direction of $\boldsymbol{\gamma}$.

2) If $(\mathrm{d}x^i)$ is a time-like 4-vector, $\mathrm{d}T$ has the same sign as $\mathrm{d}x^4$.

3) In the case that $\gamma_\alpha = 0$, $\gamma_4 = -1$, (as occurs in Special Relativity with use of lorentzian co-ordinates) $\mathrm{d}T$ becomes equal to $\mathrm{d}t$.

This definition, which follows from a geometrical operation linked to the physical system of reference and which is invariant in an internal transformation of co-ordinates, evidently has a physical meaning in the system of reference. The same meaning is obviously not enjoyed by the co-ordinate time $dt$.

In this manner we have justified the name of metric time vector, relative to the system $S$, which we gave in Sect. 1·2 to the 4-vector $\boldsymbol{\gamma}$, which, through (65), permits the calculation of the relative standard time.

Similarly for the spatial tensor $\gamma_{\alpha\beta}$, which we have already mentioned in Sect. 1·2 and which permits the calculation of the spatial norm of a 4-vector through an operation which is connected to the reference fluid but not to the co-ordinates chosen in it, the name of *standard metric space tensor relative to* $S$ is justified. The corresponding positive definite quadratic form

$$d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{67}$$

will be called, as we said in Sect. 1·2, *standard space metric, relative to* $S$.

By means of the two metrics we have introduced, the space-time metric (1) can be expressed

$$-c^2\, d\tau^2 = ds^2 = d\sigma^2 - c^2\, dT^2\,. \tag{68}$$

From these definitions there follow almost automatically the definitions of the other relative kinematic quantities. Precisely, we shall call *standard relative velocity* the 3-vector having the components

$$v^\alpha = \frac{dx^\alpha}{dT}\,. \tag{69}$$

Its spatial norm $v^2$ and its covariant components $v_\alpha$ will obviously be calculated by means of the metric space tensor $\gamma_{\alpha\beta}$:

$$v^2 = \gamma_{\alpha\beta} v^\alpha v^\beta\,, \tag{70}$$

$$v_\alpha = \gamma_{\alpha\beta} v^\beta\,. \tag{71}$$

By means of equations (66) and (68) the mutual ratios of the time intervals $dt$, $dT$, and $d\tau$ can be easily calculated. Indeed, from (66), dividing the two sides by $dT$ we obtain

$$\frac{dt}{dT} = -\frac{1}{\gamma_4} - \frac{\gamma_\beta v^\beta}{c\gamma_4}\,, \tag{72}$$

or

$$\frac{dt}{dT} = \frac{1 + \gamma_\beta (v_\beta/c)}{\sqrt{-g_{44}}}\,. \tag{73}$$

Moreover, from (68), dividing the two sides by $c^2\,\mathrm{d}T^2$, we have

$$\frac{\mathrm{d}T}{\mathrm{d}\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \tag{74}$$

This ratio is formally identical to the lorentzian factor of Special Relativity.

With regard to the ratio $\mathrm{d}t/\mathrm{d}\tau$ which has already been introduced in Sect. 2·1 it may be written

$$\Gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}T}\,\frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{\gamma_\beta(v_\beta/c) - 1}{\gamma_4(1 - (v^2/c^2))^{\frac{1}{2}}}. \tag{75}$$

It should be noted that $\Gamma$ appears simpler when expressed with the $v$''s, as is done in (72) than it does in (28) where the $u$''s are used. In any case, the identity of the two expressions can be readily verified.

The definitions of the other relative dynamic quantities are also immediate. As in Sect. 2·1, we shall call *standard relative momentum* of the particle the 3-vector whose contravariant components are

$$p^\alpha = P^\alpha = m_0\,\frac{\mathrm{d}T}{\mathrm{d}\tau}\,v^\alpha. \tag{76}$$

These may also be written

$$p^\alpha = mv^\alpha \tag{77}$$

as soon as we define the *standard relative mass* in the following manner

$$m = m_0\,\frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}: \tag{78}$$

the relative mass is formally identical with the relative mass of Special Relativity.

It should be observed that the standard relative mass just defined differs from the relative mass $m'$ of Sect. 2·1.

The covariant components of the 3-vector relative momentum are

$$p_\alpha = \gamma_{\alpha\beta}p^\beta \tag{79}$$

and, as we remarked, are not the same as the $P_\alpha$'s, since

$$P_\alpha = g_{\alpha i}P^i = \gamma_{\alpha\beta}p^\beta + mc\gamma_\alpha. \tag{80}$$

It should be noted however that the natural decomposition (cf. Sect. 1·2) of the 4-vector $\boldsymbol{P}$ with respect to the time-like vector $\boldsymbol{\gamma}$ gives

$$\boldsymbol{P} = \boldsymbol{A} + \boldsymbol{N} \tag{81}$$

where

$$A_\alpha = mc\gamma_\alpha\,, \qquad A_4 = -P_4\gamma^4\gamma_4 = -mc\,, \tag{82}$$

$$N_\alpha = P_\alpha - mc\gamma_\alpha = p_\alpha\,, \qquad N_4 = 0\,. \tag{83}$$

It thus becomes apparent that the spatial 3-vector $p_\alpha$ which is usually defined as relative momentum (a definition which we have maintained) is none other than the projection of $\boldsymbol{P}$ on the tangent 3-space orthogonal to $\gamma$. Such a definition, because of its intrinsic character, is therefore a very natural one.

In a like manner, it is perfectly justifiable to put in direct relation the total relative energy $H$ of the particle and the fourth covariant component of the vector $\boldsymbol{A}$, and retain the same definition of Sect. 2·1

$$H = -cA_4 = -cP_4 = mc^2\gamma_4\,. \tag{84}$$

3·3. *The absolute equation of motion and its natural decomposition.* – The vectorial equation (20) of the motion of a free particle is equivalent, as we have remarked, to the system of equations (23), or to the other one (24). But neither of these systems appears to be suitable, in the general case, to put into an expressive relative form equation (20), as we shave shown with sufficient clarity in Sect. 2·1.

In order to better reach our aim, let us begin by observing that if we employ $T$ instead of $\tau$ as time variable in order to locate the particle along its time track, equation (20) may be written

$$\frac{D\boldsymbol{P}}{\mathrm{d}T} = 0\,. \tag{85}$$

Let us now operate on the 4-vector which appears on the left side of (85) the natural decomposition with respect to the direction of $\boldsymbol{\gamma}$, and apply formulae (9) (11). If we put, as usual,

$$\frac{D\boldsymbol{P}}{\mathrm{d}T} = \boldsymbol{A} + \boldsymbol{N}, \tag{86}$$

we have

$$\begin{cases} \boldsymbol{A} = \dfrac{DP_4}{\mathrm{d}T}\dfrac{\boldsymbol{\gamma}}{\gamma_4}\,, \\ \boldsymbol{N} = \dfrac{D\boldsymbol{P}}{\mathrm{d}T} - \dfrac{DP_4}{\mathrm{d}T}\dfrac{\boldsymbol{\gamma}}{\gamma_4}\,, \end{cases} \tag{87}$$

and thus

$$(88)\qquad \begin{cases} A_k = \dfrac{DP_4}{\mathrm{d}T}\dfrac{\gamma_k}{\gamma_4}, \\ N_k = \dfrac{DP_k}{\mathrm{d}T} - \dfrac{DP_4}{\mathrm{d}T}\dfrac{\gamma_k}{\gamma_4}. \end{cases}$$

Equation (85) is equivalent to the two following simultaneous equations

$$(89)\qquad \boldsymbol{A} = 0\,, \qquad \boldsymbol{N} = 0\,.$$

The first one, having regard to (87) and to the fact that $\boldsymbol{\gamma} \neq 0$, is reduced to the sole scalar equation

$$(90)\qquad \frac{DP_4}{\mathrm{d}T} = 0\,.$$

The second of equations (89), taking into account that $N_4$ is identically nil, is equivalent to the three scalar equations

$$(91)\qquad N_\alpha = \frac{DP_\alpha}{\mathrm{d}T} - \frac{DP_4}{\mathrm{d}T}\frac{\gamma_\alpha}{\gamma_4} = 0\,.$$

Equation (90) is none other than the fourth one of (24), except for the substitution of $\tau$ with $T$, and, on account of (84), may be interpreted, as it usually is, as the equation of energy.

Concerning (91)'s, these do not coincide with the first three of (24)'s, but are combinations of the latter (again substituting $T$ to $\tau$) with (90).

Since the term by which they differ is nil because of (90), their equivalence is evident. Nevertheless, the (91)'s are preferable for the purpose of giving the equations of motion the desired relative interpretation.

3·4. *Standard relative equations of motion of a free particle.* – Let us consider in the first place equations (91) and render explicit the operation $D$ in conformity with (24). We have

$$(92)\qquad \frac{\mathrm{d}P_\alpha}{\mathrm{d}T} - \frac{1}{2}\,\partial_\alpha g_{\lambda\mu} P^\lambda v^\mu - mc\,\partial_\alpha g_{4\mu} v^\mu \frac{\mathrm{d}t}{\mathrm{d}T} - \frac{1}{2}\,mc^2\,\partial_\alpha g_{44}\left(\frac{\mathrm{d}t}{\mathrm{d}T}\right)^2 -$$

$$- \frac{\mathrm{d}P_4}{\mathrm{d}T}\frac{\gamma_\alpha}{\gamma_4} + \frac{1}{2}\,\partial_4 g_{\lambda\mu} P^\lambda v^\mu \frac{\gamma_\alpha}{\gamma_4} + mc\,\partial_4 g_{4\mu} v^\mu \frac{\gamma_\alpha}{\gamma_4}\frac{\mathrm{d}t}{\mathrm{d}T} + \frac{1}{2}\,mc^2\,\partial_4 g_{44}\frac{\gamma_\alpha}{\gamma_4}\left(\frac{\mathrm{d}t}{\mathrm{d}T}\right)^2 = 0\,.$$

In the formula we have just written, let us substitute for $P_\alpha$, $P^\lambda$, $P_4$ and $\mathrm{d}t/\mathrm{d}T$ their expressions (76), (80) and (72). Developing the derivatives

$\mathrm{d}(m\gamma_\alpha)/\mathrm{d}T$ and $\mathrm{d}(m\gamma_4)/\mathrm{d}T$, we have

$$(93)\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}T}+\bar{m}c\frac{\mathrm{d}\gamma_\alpha}{\mathrm{d}T}-mc\frac{\mathrm{d}\gamma_4}{\mathrm{d}T}\frac{\gamma_\alpha}{\gamma_4}-\frac{1}{2}m\left(\partial_\alpha g_{\lambda\mu}-\frac{\gamma_\alpha}{\gamma_4}\partial_4 g_{\lambda\mu}\right)v^\lambda v^\mu+$$

$$+mc\left(\partial_\alpha g_{4\mu}-\frac{\gamma_\alpha}{\gamma_4}\partial_4 g_{4\mu}\right)\left(\frac{1}{\gamma_4}+\frac{\gamma_\lambda v^\lambda}{c\gamma_4}\right)v^\mu-$$

$$-\frac{1}{2}mc^2\left(\partial_\alpha g_{44}-\frac{\gamma_\alpha}{\gamma_4}\partial_4 g_{44}\right)\left(\frac{1}{\gamma_4^2}+\frac{2\gamma_\mu v^\mu}{c\gamma_4^2}+\frac{\gamma_\lambda\gamma_\mu v^\lambda v^\mu}{c^2\gamma_4^2}\right)=0\,.$$

If we observe that

$$(94)\qquad \frac{\mathrm{d}\gamma_\alpha}{\mathrm{d}T}-\frac{\gamma_\alpha}{\gamma_4}\frac{\mathrm{d}\gamma_4}{\mathrm{d}T}=\gamma_4\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\gamma_\alpha}{\gamma_4}\right),$$

and we introduce the operation $\tilde{\partial}$ defined in Sect. 3·1 the above equations may be written

$$(95)\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}T}+mc\gamma_4\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\gamma_\alpha}{\gamma_4}\right)-\frac{1}{2}m\,\tilde{\partial}_\alpha g_{\lambda\mu}v^\lambda v^\mu+mc\,\tilde{\partial}_\alpha g_{4\mu}v^\mu\left(\frac{1}{\gamma_4}+\frac{\gamma_\lambda v^\lambda}{c\gamma_4}\right)-$$

$$-\frac{1}{2}mc^2\,\tilde{\partial}_\alpha g_{44}\left(\frac{1}{\gamma_4^2}+\frac{2\gamma_\mu v^\mu}{c\gamma_4^2}+\frac{\gamma_\lambda\gamma_\mu v^\lambda v^\mu}{c^2\gamma_4^2}\right)=0\,.$$

On the other hand, since the operator $\tilde{\partial}_\alpha$ enjoys, obviously, the same formal properties as the ordinary operator $\partial_\alpha$, having in mind (60) we find that

$$(96)\qquad \tilde{\partial}_\alpha(\gamma_\lambda\gamma_\mu)=\tilde{\partial}_\alpha\left(\frac{g_{\lambda 4}g_{4\mu}}{-g_{44}}\right)=\frac{1}{\sqrt{-g_{44}}}\tilde{\partial}_\alpha g_{\mu 4}\cdot\gamma_\mu+\frac{1}{\sqrt{-g_{44}}}\tilde{\partial}_\alpha g_{\mu 4}\cdot\gamma_\lambda+\frac{1}{g_{44}}\tilde{\partial}_\alpha(-g_{44})\cdot\gamma_\lambda\gamma_\mu\,,$$

and therefore

$$(97)\qquad \frac{1}{2}m\,\tilde{\partial}_\alpha(\gamma_\lambda\gamma_\mu)v^\lambda v^\mu=\frac{m}{\sqrt{-g_{44}}}\tilde{\partial}_\alpha g_{\lambda 4}\gamma_\mu v^\lambda v^\mu-\frac{1}{2}\frac{m}{g_{44}}\tilde{\partial}_\alpha g_{44}\gamma_\lambda\gamma_\mu v_\lambda v_\mu\,.$$

The two terms on the right side of (97) are both present, with opposite signs, in (95), which may thus be written

$$(98)\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}T}+mc\gamma_4\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\gamma_\alpha}{\gamma_4}\right)-\frac{1}{2}\tilde{\partial}_\alpha(g_{\lambda\mu}+\gamma_\lambda\gamma_\mu)p^\lambda v^\mu+$$

$$+\frac{mc}{\gamma_4}\tilde{\partial}_\alpha g_{4\mu}v^\mu-\frac{1}{2}\frac{mc^2}{\gamma_4^2}\tilde{\partial}_\alpha g_{44}-\frac{mc}{\gamma_4^2}\tilde{\partial}_\alpha g_{44}\gamma_\mu v^\mu=0\,.$$

Let us now observe that, also because of (72), we have

$$\gamma_4 \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\gamma_\alpha}{\gamma_4}\right) = \gamma_4 \tilde{\partial}_\mu \left(\frac{\gamma_\alpha}{\gamma_4}\right) v^\mu - c\,\partial_4 \left(\frac{\gamma_\alpha}{\gamma_4}\right). \tag{99}$$

At the same time we note that

$$\frac{\tilde{\partial}_\alpha g_{4\mu}}{\gamma_4} - \frac{\gamma_\mu \tilde{\partial}_\alpha g_{44}}{\gamma_4^2} = -\gamma_4 \tilde{\partial}_\alpha \left(\frac{\gamma_\alpha}{\gamma_4}\right). \tag{100}$$

Having regard to the two identities (99) and (100), equation (98), grouping the different terms in a suitable manner, becomes

$$\left(\frac{\mathrm{d}p_\alpha}{\mathrm{d}T} - \frac{1}{2}\tilde{\partial}_\alpha \gamma_{\lambda\mu} p^\lambda v^\mu\right) + mc\gamma_4 \left[\tilde{\partial}_\beta \left(\frac{\gamma_\alpha}{\gamma_4}\right) - \tilde{\partial}_\alpha \left(\frac{\gamma_\beta}{\gamma_4}\right)\right] v^\beta - mc^2 \partial_4 \left(\frac{\gamma_\alpha}{\gamma_4}\right) - \frac{1}{2}\frac{mc^2}{\gamma_4^2}\tilde{\partial}_\alpha g_{44} = 0\,.$$

Finally, if we observe that the quantity in round brackets is the transverse absolute derivative of the 3-vector $p_\alpha$, and that the quantities within square brackets coincide with the components $\Omega_{\lambda\alpha}$ of the skew-symmetrical tensor introduced in Sect. 1·3, and if we transform in an obvious manner the last term, we find

$$\frac{\tilde{\mathrm{d}}^* p_\alpha}{\mathrm{d}T} = m\left[-\frac{1}{2}c^2 \tilde{\partial}_\alpha \log\left(-g_{44}\right) + c\gamma_4 \Omega_{\alpha\beta} v^\beta + c^2 \partial_4 \left(\frac{\gamma_\alpha}{\gamma_4}\right)\right]. \tag{101}$$

The quantity within square brackets furnishes the gravitational field relative to the physical system of reference we have chosen; in it we can distinguish three terms which may be interpreted as follows:

1) The first one represents a field which derives from the scalar potential $-\frac{1}{2}c^2 \log\left(-g_{44}\right)$. If the latter is independent from $x^4$, this field is conservative.

2) The second term represents a field similar to Coriolis' apparent field in newtonian mechanics. It disappears if, and only if, the time tracks of the reference particles constitute a normal congruence.

3) The third term is a field analogous to that which is present in an electro-magnetic field when the potential 3-vector, which here is represented by the spatial vector $\gamma_\alpha/\gamma_4$, is non stationary.

The partial or approximate results, which have been quoted in Sect. 2·1, have thus been extended, without limitations or approximations, to the general case. The analogies between equations (41) and (101) are evident. But some differences should also be emphasized: the scalar potential $-\chi$ is now

replaced by the more general scalar potential $-\frac{1}{2}c^2 \log(-g_{44})$; the approximate tensor of rotation $\omega_{\alpha\beta}$ is here replaced by the tensor $\Omega_{\alpha\beta}$; finally, the approximate vector potential $\gamma_\alpha$ is replaced by $\gamma_\alpha/\gamma_4$.

We need hardly add that when the second powers and the products of the quantities $\gamma_\alpha/\gamma_4$, $\chi$, $v_\alpha/c$ and their derivatives can be neglected, (101) becomes identical with (41).

We shall examine in another paper the relations between the equations of motion (101) and the equation of energy (90).

## RIASSUNTO

In questo lavoro si mostra come alle equazioni del moto di una particella libera di prova in Relatività generale, riferite a un generico sistema fisico di riferimento, possa sempre darsi l'interpretazione classica:

*Derivata temporale dell'impulso = massa × campo gravitazionale.*

Lo scopo si raggiunge definendo opportunamente le varie grandezze dinamiche relative al riferimento prescelto, e mediante l'uso sistematico di una operazione differenziale che generalizza l'ordinaria differenziazione assoluta. Nel campo gravitazionale risultano presenti tre parti: una derivante da un potenziale scalare, una analoga al campo apparente del Coriolis, e una parte proveniente da un potenziale vettore.

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# On the Internal Conversion of the $E3$-Transition in $^{107}$Ag.

H. Schneider, I. J. van Heerden and D. Reitmann

*Nuclear Physics Division of the National Physical Research Laboratory, C.S.I.R. - Pretoria*

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**Summary.** — Measurements of the relative internal conversion coefficients of the 93.9 keV transition in $^{107}$Ag are described. The $L_1 : L_{11} : L_{111}$ and $K : L$ ratios presented are in excellent agreement with the theoretical values obtained from the tables of Rose. The $K : M$ ratio, however, is inconsistent; this is ascribed to the screening effect not being taken into account in the calculations.

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In connection with the study of finite nuclear size- and screening-effects on internal conversion, the conversion electrons from the 44.2 s isomeric transition in $^{107}$Ag were measured with an improved technique. This transition, first described by Alvarez, Helmholz and Nelson [1], is known to be an $E3$ [2], this assignment being in agreement with earlier measurements [3,4]. However, all previously available data deal with the total conversion coefficient, the $K/L$ and $K/(M+N)$ ratios and the lifetime of the isomer. The $L_1$, $L_{11}$, $L_{111}$ lines have not yet been resolved, although their relative internal conversion coefficients are very sensitive with respect to the multipolarity of the radiation.

A target of normal silver (51.4% $^{107}$Ag and 48.6% $^{109}$Ag) was irradiated for 22 hours with 7.5 MeV protons at 50 μA in the C.S.I.R. cyclotron. The 6.7 hour $^{107}$Cd and the 470 day $^{109}$Cd are produced by (p, n) reactions. The $^{107}$Cd decays

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(1) L. W. Alvarez, A. C. Helmholz and E. Nelson: *Phys. Rev.*, **57**, 660 (1940).

(2) M. Goldhaber and A. W. Sunyar: *Phys. Rev.*, **83**, 906 (1951).

(3) H. Bradt, P. C. Gugelot, O. Huber, H. Medicus, P. Preiswerk, P. Scherrer and R. Steffen: *Helv. Phys. Acta*, **20**, 153 (1947).

(4) G. E. Valley and R. L. McCreary: *Phys. Rev.*, **56**, 863 (1939).

mainly by $K$-capture to $^{107m}$Ag. The radioactive Cd was chemically and electrolytically separated from the silver target by a method described by BRADT *et al.* (3). Finally, the Cd was deposited on a 8 μm thick platinum wire by electrolysing at 6 V 40 mA for 15 minutes. This wire was then used as a source in a semi-circular permanent magnet β-ray spectrograph. Freshly poured Ilford G5 emulsion was used as detector, and the individual electron tracks counted in a microscope. This experimental procedure has been described in a previous article (5).

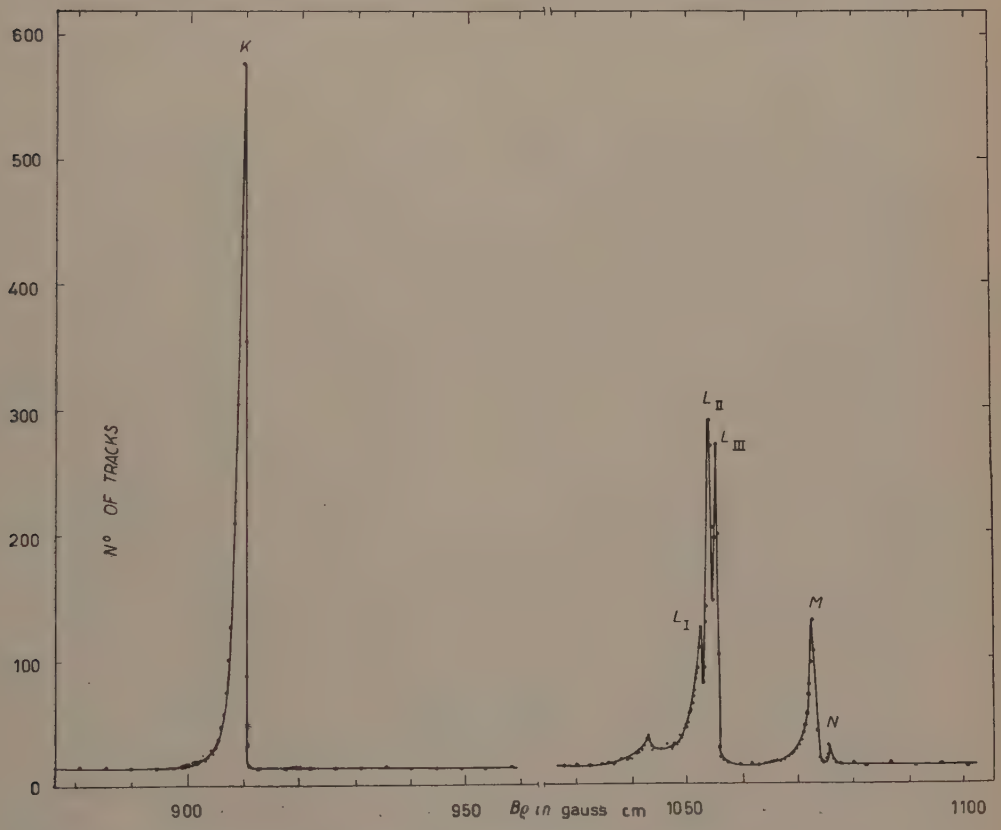


Fig. 1. – Internal conversion lines in $^{107}$Ag. Values given are uncorrected for transmission.

Fig. 1 shows the conversion lines as measured in an emulsion plate exposed for 3 hours; the actual experimental values, uncorrected for transmission, are given. Each point represents the total for 15 adjacent scans along the length

(5) D. REITMANN, H. SCHNEIDER and I. J. VAN HEERDEN: *Phys. Rev.*, **110**, 1093 (1958).

of the plate. The background is probably caused by scattered radiation, and is more or less constant along the plate. The width of the measured lines is mostly due to the finite thickness of the source and not to resolution power of the spectrograph or natural line-width. The shapes of the lines should therefore be similar, and this made it possible to separate the overlapping parts of the $L_{\mathrm{I}}$, $L_{\mathrm{II}}$ and $L_{\mathrm{III}}$ lines, as shown in Fig. 2. The small peaks appearing

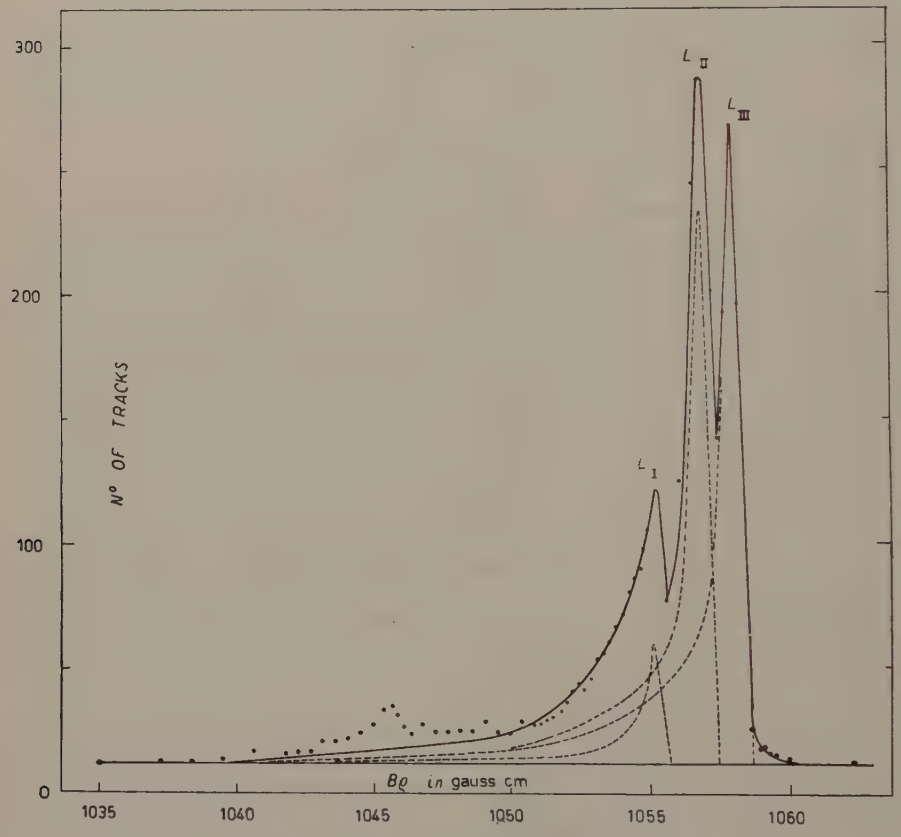


Fig. 2. – $L_{\mathrm{I}}$, $L_{\mathrm{II}}$ and $L_{\mathrm{III}}$ internal conversion lines in $^{107}$Ag.

above the low energy tails of the $L$-lines, result from $M$ and $N$ internal conversion of the 89 keV transition in $^{109m}$Ag; this is borne out by their $B\varrho$-values. Due to their low intensities, no attempt was made to measure the other internal conversion lines of this transition in this particular plate. For the sake of clearness no errors are indicated in the figures. From the above measurements, corrected for transmission, the ratios given in Table I were obtained.

The errors quoted are of a statistical nature only, and the theoretical values were obtained by interpolation from the tables of ROSE (6).

TABLE I.

| Experimental values | Theoretical values for $E3$ |
|---|---|
| $K/L = 1.08 \pm 0.03$ | 1.10 |
| $L_I/L_{II}/L_{III} = (0.18 \pm 0.01)/(0.90 \pm 0.03)/(1.00 \pm 0.02)$ | 0.18/0.85/1.00 |
| $K/M = 4.67 \pm 0.18$ | 2.21 |
| $K/(M+N) = 4.36 \pm 0.16$ | |
| $K/N = 66 \pm 9$ | |

As is shown, the $K/L$ and $L_I/L_{II}/L_{III}$ ratios agree very well with the theoretical values for an $E3$ transition, whereas the $K/M$ value is too large by a factor of about two. The value for $K/(M+N)$ is however in substantial agreement with values obtained previously (3). As the finite nuclear size-effect is expected to be negligibly small at $Z = 47$, the observed discrepancy is very probably due to the fact that the screening-effect has not been taken into account in the theoretical calculations for the $M$-shell. As stated by ROSE this effect may decrease some of the calculated $M$-shell coefficients by as large a factor as two.

* * *

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(6) M. E. ROSE: privately circulated tables.

## RIASSUNTO (*)

Si descrivono misure dei coefficienti relativi di conversione interna per la transizione di 93.9 keV in $^{107}Ag$. I rapporti $L_I : L_{II} : L_{III}$ e $K : L$ osservati sono in eccellente accordo coi valori teorici ottenuti dalle tabelle di Rose. Il rapporto $K : M$, invece, è incompatibile; ciò è imputabile al fatto di non aver tenuto conto nei calcoli dell'effetto di schermo.

(*) *Traduzione a cura della Redazione.*

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# On $\pi \to e + \nu + \gamma$ Decay.

V. G. Vaks and B. L. Ioffe

*Academy of Sciences - Moscow*

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**Summary.** — The $\pi \to e + \nu + \gamma$ decay is considered for the case of vector and axial-vector interaction. The assumption that in the vector-variant there is a direct interaction of $\pi$-mesons with an electron-neutrino field as proposed by Gell-Mann and Feynman presents the possibility of establishing an exact relation between the probability of $\pi \to e + \nu + \gamma$ decay, through the vector-variant and the probability of $\pi^0 \to 2\gamma$ decay. As a result, it appears that in the total probability of $\pi \to e + \nu + \gamma$ decay the main role is played by decay through the axial variant, and the ratio of total probability of $\pi \to e + \nu + \gamma$ decay to the probability of $\pi \to \mu + \nu$ decay is of the order of $5 \cdot 10^{-6}$. Expressions for the angular and energy distribution of electrons and quanta have been obtained.

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Gell-Mann and Feynman [1] proposed a scheme of universal weak interaction in which the interaction of nucleons with the electron-neutrino field is performed with the aid of the vector and axial vector variants, so that the Hamiltonian interaction has the form of

$$H_1 = (\bar{\psi}\gamma_\mu(G_v + \gamma_5 G_A)\tau^+\psi)I_\mu + \text{herm. conj.}$$

$$I_\mu = \left(\bar{\psi}_e \gamma_\mu \frac{1+\gamma_5}{2}\psi_\nu\right), \tag{1}$$

where $\psi = \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}$ are wave functions of heavy particles,

$$\tau^+ = \sqrt{2}\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} = \frac{1}{\sqrt{2}}(\tau_x + i\tau_y),$$

$$\gamma_5 = -\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \gamma_\mu = \{\beta, \beta\boldsymbol{\alpha}\},$$

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[1] M. Gell-Mann and R. Feynman: *Phys. Rev.*, **101**, 193 (1956).

$G_V$ and $G_A$ are coupling constants. GELL-MANN and FEYNMAN also assumed that there is a direct interaction of $\pi$-mesons with an electron-neutrino field, described by the Hamiltonian

$$H_2 = 2iG_V[\mathcal{P}^+ T^+ \nabla_\mu \mathcal{P} - (\nabla_\mu \mathcal{P}^+) T^+ \mathcal{P}] I_\mu + \text{herm. conj.} \tag{2}$$

where $\mathcal{P} = \{\varphi^-\, \varphi^0, \varphi^+\}$ are wave functions of $\pi$-mesons

$$T^+ = \frac{1}{\sqrt{2}}(T_x + iT_y) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

is an isotopic spin operator of $\pi$-mesons. The vector part of the interaction (1) together with interaction (2) is fully analogous to the interaction of nucleons and $\pi$-mesons with the electromagnetic field

$$H_{el} = e\{\bar{\psi}\gamma_\mu(\tfrac{1}{2}\tau_z + \tfrac{1}{2})\psi + i[\mathcal{P}^+ T_z \nabla_\mu \mathcal{P} - (\nabla_\mu \mathcal{P}^+) T_z \mathcal{P}]\} A_\mu . \tag{3}$$

That permits the establishment of an exact correlation between the probability of the $\pi^\pm \to e^\pm + \nu + \gamma$ decay, which takes place through the vector variant and the decay probability of the neutral $\pi$-meson into two $\gamma$-quanta. Since during the $\pi^\pm \to e^\pm + \nu + \gamma$ decay, which takes place through the axial-vector variant, the main role is played by terms which are simply connected with the $\pi^\pm \to e^\pm + \nu$ decay, there thus appears the possibility of a more correct evaluation of the probability of the $\pi^\pm \to e^\pm + \nu + \gamma$ decays in the scheme with $A$ and $V$ variants than was made previously [2].

Let us write down the matrix elements of the processes $\pi^0 \to 2\gamma$, $M_\gamma$ and $\pi \to e + \nu + \gamma$ through the vector variant $M_V$ in the form

$$\begin{cases} M_\gamma(k_1 k_2) = e^2 \varphi^0(q) A_\mu(k_1) A_\nu k_2)(\, U^\gamma_{\mu\nu}(k_1, k_2)\,, \\ M_V(k_1 k_2) = eG_V \varphi^+(q) A_\mu(k_1) I_\nu(k_2)\, U^V_{\mu\nu}(k_1 k_2)\,. \end{cases} \tag{4}$$

Here, in the case of $\pi^0 \to 2\gamma$ decay $k_1$ and $k_2$ are 4-momentums of quanta, $\varphi^0(q)$ is the wave function of the neutral meson, $q$ is its momentum $q = k_1 + k_2$; in the case of $\pi \to e + \nu + \gamma$ decay $k_1$ is the $\gamma$-quantum momentum, $k_2$ is the sum of momenta of the electron and the neutrino $k_2 = p_e + p_\nu$, $\varphi(q)$ is the wave function of the charged meson.

We shall show that when strong interactions are accurately taken into account (neglecting radiative corrections connected with electromagnetic and

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[2] S. B. TREIMAN and H. W. WYLD: *Phys. Rev.*, **101**, 1552 (1956).

weak interactions), then

$$U^{\gamma}_{\mu\nu}(k_1k_2) = U^{V}_{\mu\nu}(k_1k_2)\,. \tag{5}$$

We shall suppose that the momenta of the emitted $\gamma$-quanta and leptons ($k \sim \mu/2$) are small in comparison with the average momenta $\Lambda$ of the virtual particles through which the decay takes place (*).

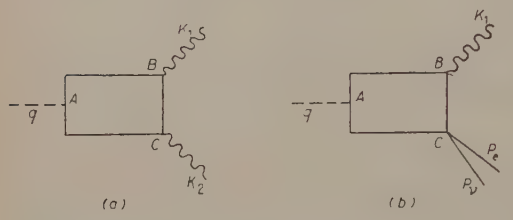


Fig. 1. – *a*) $\pi^0 \to 2\gamma$ decay; ------ the meson line, ——— the photon line; *b*) $\pi^- \to e + \nu + \gamma$ decay; ——— electron and neutrino lines; $p_e$ and $p_\nu$ are electron and neutrino 4-momenta.

In the beginning we shall confine ourselves to the case in which in strong interactions participate only $\pi$-mesons and nucleons; let us consider the arbitrary Feynman diagram (Fig. 1) for the processes which interest us. Values $U^{\gamma}_{\mu\nu}(k_1k_2)$ and $U^{\nu}_{\mu\nu}(k_1k_2)$ for this diagram are the product of two factors: 1) the integral over the momenta of virtual particles and the trace over spin variables and 2) a trace along isotopic variables. A comparison of interaction Hamiltonians (1) and (2) with (3) shows that the spinor-momentum part of the diagram is the same for $U^{\gamma}_{\mu\nu}$ and $U^{\nu}_{\mu\nu}$. Thus $U^{\gamma}_{\mu\nu}$ and $U^{\nu}_{\mu\nu}$ have the form

$$\begin{cases} U^{\gamma,V}_{\mu\nu}(k_1, k_2) = I_{\mu\nu}(k_1, k_2) F^{\gamma,V}_{ABC}\,, \\ F^{\gamma,V}_{ABC} = \mathrm{tr}\,\{\tau_A \dots \tau_B \dots \tau_C \dots\}\,, \end{cases} \tag{6}$$

where $\tau_A$, $\tau_B$, $\tau_C$ are isotopic operators in which the index $A$ corresponds to the initial $\pi$-meson while indexes $B$ and $C$ correspond to two $\gamma$-quanta in $\pi^0$-decay and to a $\gamma$-quantum and a lepton current in $\pi \to e + \nu + \gamma$ decay. In the case of $\pi^0$-decay $\tau_A$ must be substituted by $\tau_z$, while $\tau_B$ and $\tau_C$ during the emission of $\gamma$-quanta by nucleons by $(1+\tau_z)/2$ and during the emission of $\gamma$-quanta by $\pi$-mesons by $T_z$. For the $\pi^- \to e^- + \nu + \gamma$ decay instead of $\tau_A$ we have $\tau = 1/\sqrt{2}\,(\tau_x - i\tau_y)$, instead of $\tau_B$ we have $(\tau_z+1)/2$ (or $T_z$) and, instead of $\tau_C$, $\tau^+$ (or $T^+$).

The quantity $I_{\mu\nu}(k_1, k_2)$ must be a pseudo-tensor of the second rank, *i.e.*

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(*) From the experimenrally determined life-time of a $\pi^0$-meson it follows that $\Lambda$ cannot be substantially less than the $M$-mass of a nucleon since if $\Lambda/M \ll 1$ in the evaluation of the life-time of a $\pi^0$-meson there would arise an additional factor $(\Lambda/M)^4$ which would cause a contradiction with the experiment. (This assertion does not make use of the perturbation theory since the other terms of the expansion in $g^2$ would be of the order of $g^2(\Lambda/M)^4 \lesssim 1$ or less).

should have the form

$$I_{\mu\nu}(k_1, k_2) = A\,\varepsilon_{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}\,, \tag{7}$$

where $A$ does not depend on $k_1$ and $k_2$ disregarding the terms of the order of $(k/A)^2$. Alongside with the diagram of Fig. 1 we shall now consider the diagram in which vertices $B$ and $C$ change places. It is not difficult to see that due to the symmetry (7) regarding the substitution $\mu \to \nu$, $k_1 \to k_2$ the sum of these diagrams is equal to

$$U^{\gamma,V}_{\mu\nu}(k_1, k_2)\; U^{\gamma,V}_{\nu\mu}(k_2, k_1) = I_{\mu\nu}(k_1 k_2)[F^{\gamma,V}_{ABC} + F^{\gamma,V}_{ACB}]\,. \tag{8}$$

We shall elucidate the vector structure of the expression $F_{ABC} + F_{ACB}$ in the isotopic space. For this we note that interaction with the electromagnetic field (since we do not take into consideration radiation corrections in $e^2$), may be formally described as a sum of interactions of nucleons and $\pi$-mesons with the isotopic vector $\alpha^\gamma_i$ which has components $\alpha^\gamma_z = \frac{1}{2}$, $\alpha^\gamma_z = \alpha^\gamma_y = 0$ and the isotopic scalar $\beta = \frac{1}{2}$. Then the Hamiltonian of interaction with the electromagnetic field has the following expression:

$$H_{ei} = e\{\bar{\psi}\gamma_\mu \tau_i \psi + i[\mathcal{P}^+ \nabla_\mu T\mathcal{P} - (\nabla_\mu \mathcal{P}^+) T_i \mathcal{P}]\} A_\mu \alpha^\gamma_i + e\bar{\psi}\gamma_\mu \psi A_\mu \beta$$

and will formally have an isotopically invariant form. The espression $F^\gamma_{ABC}$ will also take on an isotopically invariant form

$$F^\gamma_{ABC} = F_{ikl}\varphi^0_i \alpha^\gamma_k \alpha^\gamma_l + F'_{ik}\varphi^0_i \alpha^\gamma_k \beta + F''_{il}\varphi^0_i \beta \alpha^\gamma_l + F'''_i \varphi^0_i \beta^2\,, \tag{9}$$

where $\varphi^0_i$ is an isotopic vector of a $\pi^0$-meson ($\varphi^0_1 = \varphi^0_2 = 0$, $\varphi^0_3 = 1$). The term $F'_{ik}$ corresponds to the case when in the vertex $B$ $\alpha^\gamma_k$ is responsible for the emission of a quantum while $\beta$ is responsible for it at the vertex $C$ and the term $F''_{il}$ holds good for the opposite case. Similarly the interaction with the lepton field may be written down in the form of an interaction with the isotopic vector $\alpha^V_l$ with components $\alpha^V_z = 0$, $\alpha^V_x = 1/\sqrt{2}$, $\alpha^V_y = i/\sqrt{2}$. In this case expressions for $F^V_{ABC}$ and $F^V_{ACB}$ will have the form ($\varphi^+_i$ isotopic vector of a $\pi^-$-meson, $\varphi_1 = 1/\sqrt{2}$, $\varphi_2 = -i/\sqrt{2}$, $\varphi_3 = 0$)

$$\begin{cases} F^V_{ABC} = F_{ikl}\varphi_i \alpha^\gamma_k \alpha^V_l + F''_{il}\varphi_i \beta \alpha^V_l\,, \\ F^V_{ACB} = F_{ikl}\varphi_i \alpha^V_k \alpha^\gamma_l + F'_{ik}\varphi_i \alpha^V_k \beta\,, \end{cases} \tag{9$^1$}$$

since the isotopic scalar $\beta$ participates only in the emission of a quantum, (*i.e.* at the vertex $B$ of the diagram of Fig. 1). It can be easily seen that those terms in the expression $F^{\gamma,V}_{ABC} + F^{\gamma,V}_{ACB}$ in which both interactions at vertices

$B$ and $C$ take place at the expense of the isotopic vector, vanish. Indeed, in this case in (9) and (9¹) the value $F_{ikl}$ should be a pseudo-tensor of the third rank in the isotopic space. However, after summing over isotopic variables of mesons and taking the trace over the isotopic variables of nucleons, the sole expression by which $F_{ikl}$ may be expressed is the single tensor $\varepsilon_{ikl}$. But $\varepsilon_{ikl}$ is antisymmetrical over the $k$, $l$ indexes while $F_{ikl}+F_{ilk}$ is symmetrycal over these indexes. It follows from above that $F_{ikl}+F_{ilk}=0$ (so that, for example, both $\gamma$-quanta in the $\pi^0$-decay cannot be emitted by the virtual $\pi$-mesons).

Similarly vanish those terms $F^{\gamma}_{ABC}$ in which both emissions of a $\gamma$-quantum at vertexes $B$ and $C$ take place at the expense of an isotopic scalar. Indeed, the isotopic vector $F^{\gamma}_{i}$ cannot be constructed, since there is no singled out direction in the isotopic space. The only terms which do not vanish are the terms $F'_{ik}$ and $F''_{il}$ in (9) and (9¹). The quantities $F'_{ik}$ and $F''_{ik}$ are tensors of the second rank in the isotopic space and, therefore, may be expressed by a single tensor $\delta_{ik}$, *i.e.* $F'_{ik}=C\delta_{ik}$, $F''_{ik}=D\delta_{ik}$, where $C$ and $D$ are certain numerical constants. Substituting these expressions for $F'_{ik}$ and $F''_{ik}$ in (9) and (9¹), we easily find that $F^{\gamma}_{ABC}+F^{\gamma}_{ACB}=F^{\nu}_{ABC}+F^{\nu}_{ACB}=\frac{1}{2}(C+D)$ which proves the relationship (5).

The given proof is easily generalized to cases when hyperons and K-mesons also participate in strong interactions. In this case in the Hamiltonian of the interaction of hyperons and K-mesons with an electromagnetic field should be written down instead of matrices $(\tau_z+1)/2$ and $T_z$ corresponding matrices $T_z+\frac{1}{2}+(s/2)$ and $T_z+(s/2)$, where $s$ is the strangeness, and the assumption should be made, that the Hamiltonians of interaction of hyperons and K-mesons with the lepton field have the form (1), (2), where instead of $\tau^+$ and $T^+$ there is the corresponding projection of the operator of the isotopic spin of the strange particle.

With the aid of (4), (5) and (7) we may write down the matrix elements $M_{\gamma}$ and $M_{\nu}$ in the form:

$$(10)\qquad \begin{cases} M_{\gamma}(k_1, k_2) = ae^2\varphi^0(q)A_{\mu}(k_1)A_{\nu}(k_2)\varepsilon_{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma} \\ M_{\nu}(k_1, k_2) = aeG_{\nu}\varphi(q)A_{\mu}(k_1)I_{\nu}(k_2)\varepsilon_{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}, \end{cases}$$

and determine the constant $a$ from the experimentally measured life-time of a $\pi^0$-meson.

The matrix element $M_A(k_1, k_2)$ corresponding to the $\pi \to e+\nu+\gamma$ decay through the axial-vector variant may conveniently be divided into two terms. To the first term is referred the diagram (Fig. 2) in which a $\gamma$-quantum is emitted by an electron and also the terms of the zero power in $k_1$, $k_2$ in expanding the diagram of Fig. 1 to the power $k$. Let us refer to the quadratic

expressions in $k_1$ and $k_2$ of the second term which appear during the expansion to the power $k$ of the diagram of Fig. 1.

As is known [2-4] for the determination of the first term $M_A(k_1, k_2)$ there is no necessity for computing closed loops. It can be obtained if a phenomenological interaction is introduced:

$$H_{\pi e\nu} = g_{\pi e\nu} \frac{\partial \varphi^+}{\partial x_\mu} \left( \overline{\psi}_e \gamma_\mu \frac{1+\gamma_5}{2} \psi_\nu \right) + \text{herm. conj.} \tag{11}$$

describing the $\pi \to e + \nu$ decay and if one considers that during the $\pi \to e + \nu + \gamma$ decay due to the gradient invariance $\partial\varphi^+/\partial x_\mu$ shall be substituted by $\partial\varphi^+/\partial x_\mu + ieA_\mu$ (expression $ieg_{\pi e\nu} A_\mu \varphi^+ I_\mu$ corresponds to terms of zero order in the $k_1$, $k_2$ expansion of the diagram of Fig. 1). Then [2-4] Hamiltonian (11) proves to be the equivalent of Hamiltonian:

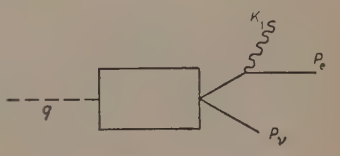


Fig. 2. – The same notations as in Fig. 1.

$$H_{\pi e\nu} = m_e g_{\pi e\nu} y \left( \overline{\psi}_e \frac{1+\gamma_5}{2} \psi_\nu \right) + \text{herm. conj.}, \tag{12}$$

where emission of $\gamma$-quanta takes place only by means of bremsstrahlung of an electron. The radiative decay of a $\pi$-meson with the Hamiltonian (12) was considered previously. Therefore, there remains for us only to analyze terms in $M_A(k_1 k_2)$ quadratic in $k_1$, $k_2$.

We can see without difficulty that the general expression for this matrix element should have the form:

$$M_A(k_1 k_2) = ibeG_A \varphi(q) [A_\mu(k_1) k_{1\nu} - A_\mu(k_1) k_{1\mu}] I_\mu(k_2) k_{2\nu} .$$

The constant $b$ strictly speaking, cannot be determined theoretically (*). However, since in the perturbation theory it is determined by integrals of the same type as the constant $a$ which appears in (10), we shall consider that $b$ is of the order of $a$.

By direct computation we can easily arrive at the fact that the matrix element of the Hamiltonian (12) does not interfere with matrix elements (10) and (13), if we disregard the higher degrees of $m_e/\mu$. Therefore, it is convenient

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[3] B. Ioffe and A. Rudik: *Dokl. Akad. Nauk SSSR*, **87**, 359 (1952).

[4] S. A. Bludman and M. A. Ruderman: *Phys. Rev.*, **101**, 910 (1956).

(*) Value $b$ has the same phase as $a$ so that they may be considered to be real. This follows from the invariance of strong and electromagnetic interactions under time inversion (or charge conjugation). Proof of this may be given with the aid of a phenomenological Hamiltonian, corresponding to matrix elements (10) and (13).

first to compute $dW^{(1)}$ the probability of $\pi \to e+\nu+\gamma$ transition at the expense of terms $M_V + M_A$.

After summing over polarizations of an emitted quantum and electron spins for the differential probability $dW^{(1)}$ we get the following expression:

$$dW^{(1)} = \frac{1}{\pi^2}\frac{C_\nu^2}{e^2} W_{\pi^0} \frac{k}{\mu^2}\left\{(1+\lambda^2)\left(1-\frac{(\boldsymbol{kn})(\boldsymbol{k\nu})}{k^2}\right)+2\lambda\frac{\boldsymbol{k\nu}-\boldsymbol{kn}}{k}\right\} \cdot \delta(E_e+E_\nu+k-\mu)\frac{d\boldsymbol{k}\,d\boldsymbol{p}}{(2\pi)^3}\,.$$

Here $\boldsymbol{p}_e = \boldsymbol{n}p_e$ and $\boldsymbol{K}$ are the momenta of electron and $\gamma$-quantum, $\nu$ is the unit vector of the direction of a neutrino momentum, $W_{\pi^0}$ the probability of decay of a $\pi^0$-meson into $2\gamma$, $\mu$ the mass of a $\pi$-meson, $\lambda = bG_A/aG_V$. $W_{\pi^0}$ is connected with a constant $a$ introduced into (10) by the relation

$$W_{\pi^0} = \pi^2\mu^3 a^2\,.$$

By integration from (14) different particular distributions can be obtained. The energy distribution of electrons has the form $(p_e = (\mu/2)y)$:

$$dW_e^{(1)} = \frac{1}{2^3\pi^3}\frac{G_\nu^2\mu^4}{e^2} W_{\pi^0} y^2\left\{(1+\lambda^2)\left[(1-y)^2+\frac{y^2}{6}\right]+\lambda(1-y)\left(1-2y+\frac{y^2}{2}\right)\right\}dy\,, \tag{15}$$

while the energy distribution of $\gamma$ quanta $(k = (\mu/2)x)$:

$$dW_\gamma^{(1)} = \frac{1}{2^6\pi^3}\frac{G_\nu^2\mu^4}{e^2}\cdot W_{\pi^0}\cdot\frac{4}{3}(1+\lambda^2)x^2(1-x)\,dx\,, \tag{16}$$

The integral probability $W^{(1)}$ is equal to

$$W^{(1)} = \frac{1}{960\pi^3}\frac{G_V^2\mu^5}{e^2} W_{\pi^0}(1+\lambda^2)\,. \tag{17}$$

Let us evaluate the numerical value of $W^{(1)}$. The constant $G_\nu^2$ was determined in experiment [5] on the $\beta$-decay of $^{14}O$: $G_V^2\mu^4 = 0.51\cdot 10^{-13}$. Evaluating the lifetime of a $\pi^0$-meson according to the work of OREAR [6]: $W_{\pi^0} \gtrsim 0.5\cdot 10^{16}\ s^{-1}$, we find (at $\lambda = 1$) $W^{(1)} = 2.4\ s^{-1}$. Thus, the ratio of $W^{(1)}$ to total probability

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[5] J. B. GERHARDT: *Phys. Rev.*, **95**, 288 (1954).
[6] G. HARRIS, Y. OREAR and S. TAYLOR: *Phys. Rev.*, **106**, 327 (1957).

of $\pi \to \mu + \nu$ decay comprises

$$\frac{W^{(1)}}{W_{\mu+\nu}} \simeq 6 \cdot 10^{-3} . \tag{18}$$

This estimation of the number of $\pi \to e + \nu + \gamma$ decays is very close to that which has been made by TREIMAN and WYLD (2) (*).

Let us consider now $W^{(2)}$ the probability of $\pi \to e + \nu + \gamma$ decay, for which are responsible those terms, where the $\gamma$-quantum is emitted by the electron. The corresponding calculation was made in (3) (+) for the case of $\pi \to \mu + \nu + \gamma$ decay, the results of which are directly transferred to $\pi \to e + \nu + \gamma$ decay. For the differential probability the following expression was obtained:

$$\mathrm{d}W^{(2)} = \frac{1}{8\pi^3} \frac{e^2}{\mu^2} W_{e+\nu} \frac{1}{E_e E_\nu k} \left\{ 2(kE_e - \boldsymbol{k}\boldsymbol{p}_e)(k\mu - kE_e + \boldsymbol{k}\boldsymbol{p}_e) + \right. \tag{19}$$

$$\left. + \mu^2 \left[ p_e^2 - \frac{(\boldsymbol{k}\boldsymbol{p}_e)^2}{k^2} \right] \right\} \frac{\mathrm{d}\boldsymbol{p}_e \, \mathrm{d}\boldsymbol{k}}{(Ek - \boldsymbol{k}\boldsymbol{p}_e)^2} \, \delta(E_e + E_\nu + k - \mu) ,$$

where $W_{e+\nu}$ is the probability of the $\pi \to e + \nu$ decay, which in the scheme of Gell-Mann and Feynman makes $1.3 \cdot 10^{-4} W_{\mu+\nu}$. The energy distribution of electrons has the following form

$$\mathrm{d}W_e^{(2)} = \frac{e^2}{2\pi} W_{e+\nu}(1-y) \left\{ \ln \left[ \left( \frac{\mu y}{m} \right)^2 \frac{1}{1-y} \right] + \frac{2y}{(1-y)^2} \left[ -2 + \ln \left( \frac{\mu y}{m} \right)^2 \right] \right\} \mathrm{d}y . \tag{20}$$

The spectrum of electrons diverges close to the upper boundary ($y \to 1$), which corresponds to the emission of long-wave quanta. Therefore, total probability of $\pi \to e + \nu + \gamma$ decay with the emission of an electron with energy, which is less than some maximal $y_{max}$ logarithmically grows with the increasing of $y_{max}$.

Due to that, also because of the presence in (20) of a large $\ln (\mu/m)^2$, the probability of radiation decay with the emission of quantum by a virtual electron has proven to be comparatively large, and considerably exceeds the probability $W_{(1)}$.

Hence, it follows that in calculating the total probability $\pi \to e + \nu + \gamma$ decay, it is sufficient to be limited by term $W^{(2)}$. Results of calculation of the

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(*) A similar computation for the case of $\pi \to \mu + \nu + \gamma$ decay gave for the probability of $\pi \to \mu + \nu + \gamma$ decay through the $V$ variant a negligible small quantity

$$W_{\mu+\nu+\gamma} \sim 5 \cdot 10^{-10} \, W_{\mu+\nu} .$$

(+) We note that in the work (3) formula (6) for the total probability of the $\pi \to \mu + \nu$ decay contains the erroneous factor $\frac{1}{2}$. In connection with this the figures of Table I of that work should be reduced by one half.

probability of $\pi \to e+\nu+\gamma$ decay as a function of the maximal energy of an electron are presented in Table I (it being accepted that $W_{e+\nu} = 1.3 \cdot 10^{-4} W_{\mu+\nu}$).

As can be seen from the data of Table I total probability of $\pi \to e+\nu+\gamma$ decay exceeds the probability of decay through a vector variant almost two orders.

Besides the energy distribution of electrons the angular distribution of quanta along an angle $\theta$ between directions of emission of an electron and a

TABLE I.

| $y_{max}$ | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| $10^6(W_{e+\nu+\gamma}/W_{\mu+\nu})$ | 0.36 | 0.85 | 1.8 | 3.3 |

quantum is also of interest. This distribution is obtained from (19) by integrating over energies of an electron and a quantum and at $\theta \gg m_e/\mu$ it proves to be equal to

$$(21) \qquad dW^{(2)}(\theta) = \frac{e^2}{4\pi} W_{e+\nu} \frac{\sin\theta\, d\theta}{\alpha^3} \cdot \left\{\alpha + (1-\alpha)\ln(1-\alpha) + 2\alpha^2(1-\alpha)\left(\ln\frac{1}{1-y} - 1\right)\right\},$$

where $\alpha = \sin^2\theta/2$ and $y_{max}$ is the maximal energy (in fractions $\mu/2$), to which the electron spectrum is integrated. Expression (21) at small $\theta$ behaves as $d\theta/\theta$. However, the divergency at $\theta \to 0$ in (21) in reality is only seeming and is conditioned by the fact that in computing (21) we disregard quantities $\sim m/\mu$. At $\theta \sim m/\mu$ instead of the denominator $1/\theta^2$ there appears an expression of the order $1/(\theta^2+4(m^2/\mu^2))$. The presence of such a factor is quite clear: it is conditioned by the fact that in a relativistic case the bremsstrahlung of the electron is concentrated in a narrow angle $\sim m/E$ near the direction of the electron momentum. This also explains the appearance in (20) of a large logarithm $\ln(\mu/m)^2$. Thus, the most favourable conditions for the disclosure of a $\pi \to e+\nu+\gamma$ decay, in the case when the decay passes through a vector and axial-vector variant, will take place only then, when the recorded electron and quantum are flying in the same direction.

The experimental observation of a decay $\pi \to e+\nu+\gamma$ by the vector variant is of special interest, since it would offer the possibility to check the Gell-Mann and Feynman hypothesis as to the presence of direct interaction of $\pi$-mesons with an electron-neutrino field and the theorem of the connection of the probabilities of decays $\pi^0 \to 2\gamma$ and $\pi \to e+\nu+\gamma$ through the vector

variant. In principle it is possible to distinguish the $\pi \to e+\nu+\gamma$ decay through the vector variant on the background of a more probable decay through the axial-vector variant, if electrons and quanta flying in opposite directions are selected, since during the decay through the vector-variant the main fraction of quanta fly out at an angle close to 180° about the direction of the emission of an electron. In this case, as follows from (21), the probability of such a decay through the axial variant, when a $\gamma$-quantum is emitted by an electron will be of the order $10^{-7} W_{\mu+\nu} \sin\theta \, d\theta$ and the probability of decay through a vector-variant will also be of the same order. At the same time, in order to single out from the probability of the transition the only terms corresponding to the vector variant (*i.e.* to determine the parameter $\lambda$ in (14)) it is necessary to have similar measurements for two different electron energies which complicates the problem which is in itself not simple (*).

* * *

In conclusion the authors wish to express their deep gratitude to I. Y. Kobzarev and L. B. Okun for a number of valuable discussions.

(*) After this work had been completed new material (7) appeared on the relation of probabilities of $\pi \to e+\nu$ and $\pi \to \mu+\nu$ decay from which it follows that $W_{e+\nu}/W_{\mu+\nu} > 10^{-5}$ Such a $W_{e+\nu}/W_{\mu+\nu}$ contradicts the Feynman and Gell-Mann theory and points to a noticeable suppression of $\pi \to e+\nu$ decay through the axial variant. At the same-time the excellent coincidence of constants of the vector variant, computed fron $\beta$-decay and $\mu$-decay is a serious argument in favour of the correctness of the Feynman and Gell-Mann theory for the vector variant. Under these circumstances the experimental disclosure of the $\pi \to e+\nu+\gamma$ decay acquires a still greater interest; due to the suppression of the axial variant it will have to go through the vector variant in the main, so that the probability of the $\pi \to e+\nu+\gamma$ decay will be determined by formulae (14)–(18).

(7) H. L. Anderson and C. M. G. Lattes: *Nuovo Cimento*, **6**, 1356 (1957).

## RIASSUNTO (*)

Si considera il decadimento $\pi \to e+\nu+\gamma$ per i casi d'interazione vettoriale e assovettoriale. L'ipotesi proposta da Gell-Mann e Feynman che nel variante vettoriale c'è interazione diretta dei mesoni $\pi$ con un campo di elettroni e neutrini, offre la possibilità di stabilire una relazione esatta fra la probabilità del decadimento $\pi \to e+\nu+\gamma$ ad opera del variante vettoriale e quella del decadimento $\pi^0 \to 2\gamma$. Si ottiene come risultato che nella probabilità totale del decadimento $\pi \to e+\nu+\gamma$ la parte principale è data dal decadimento ad opera del variante vettoriale e il rapporto della probabilità totale del decadimento $\pi \to e+\nu+\gamma$ alla probabilità del decadimento $\pi \to \mu+\nu$ è dell'ordine di $5 \cdot 10^{-6}$. Si sono ottenute espressioni per le distribuzioni angolari ed energetiche degli elettroni e dei quanti.

(*) *Traduzione a cura ella Redazione.*

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# On the Proof of the Absence of Renormalization of the Constant in the Vector-Variant of β-Interaction.

B. L. Ioffe

*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — It is proved that the vector coupling constant suffers no renormalization if, following Feynman and Gell-Mann, the β-decay hamiltonian is taken as (1) and (6).

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Gell-Mann and Feynman [1] (see also [2]) made the supposition that the constant of the vector part of β-decay interaction will not be subjected to renormalization due to interaction of mesons with nucleons, if the direct interaction between π-meson and electron-neutrino field is so introduced as to give the vector part of the Hamiltonian β-interaction of mesons and nucleons the shape of

$$(1) \qquad \begin{cases} H = G_{\nu}[\bar{\psi}\gamma_{\mu}\tau^{+}\psi + 2i\,(\Phi^{+}T^{+}\nabla_{\mu}\Phi) - 2i(\nabla\Phi^{+})T^{+}\Phi]I^{\mu}\,, \\ I_{\mu} = \tfrac{1}{2}\bar{\psi}_{e}\gamma_{\mu}(1+\gamma_{5})\psi_{\nu}\,, \end{cases}$$

where $\tau^{+}=\frac{1}{2}(\tau_{x}+i\tau_{y})$, $T^{+}=\frac{1}{2}(T_{x}+iT_{y})$ are isotopic spin operators and $\Phi=(\varphi^{-},\varphi^{0},\varphi^{+})$ meson wave functions. The assertion made by Gell-Mann and Feynman can be proved strictly, if it is noted that in the presence of β-interaction (1) the full Lagrangian of nucleons and π-mesons (accounting for their interaction, but excluding the interaction with the electromagnetic field) permits a group

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[1] M. Gell-Mann and R. Feynman: *Phys. Rev.*, **109**, 193 (1958).

[2] S. S. Gerstein and Ya. B. Zeldovič: *Žu. Èksp. Teor. Fiz.*, **29**, 698 (1955).

of infinitesimal transformations

$$(2)\qquad \begin{cases} \psi = [1 - i(\tau^+\chi + \tau\chi^+)]\psi' ; \quad \Phi = [1 - 2i(T^+\chi + T\chi^+)]\Phi' ; \\ I_\mu = I'_\mu + \dfrac{\partial\chi}{\partial x_\mu} , \end{cases}$$

with an infinitesimal numerical function $\chi$. The presence of a group of transformations (2) permits to prove a theorem analogous to Ward's theorem in quantum electrodynamics. For proof it is sufficient to calculate Green's function of the nucleon $G(x, y; I_\mu)$ in the presence of an external $\beta$ current $I_\mu$ which is independent of co-ordinates and time, and to determine the vertex part as

$$\Gamma^+_\mu(x, y; \xi) = \left.\frac{\partial G^{-1}(x, y; I_\mu)}{\partial I_\mu}\right|_{I_\mu = 0} \delta(\xi) .$$

If it is assumed $\chi(x) = I_\mu x_\mu$ then from the definition of Green's function

$$G(x, y; I_\mu) = \langle 0 \,|\, T\{\psi(x), \bar{\psi}(y)\} \,|\, 0\rangle$$

and correlation (2) it follows

$$(3)\qquad \left.\frac{\partial G(x, y; I_\mu)}{\partial I_\mu}\right|_{I_\mu = 0} = i \langle 0 \,|\, T\{\tau^+\chi_\mu\psi(x), \bar{\psi}(y)\} \,|\, 0\rangle - \\ - i \langle 0 \,|\, T\{\psi(x), \bar{\psi}(y)\tau^+ y_\mu\} \,|\, 0\rangle .$$

The value

$$\left.\frac{\partial G(x, y; I_\mu)}{\partial I_\mu}\right|_{I_\mu = 0}$$

considered as a matrix in an isotopic space must have the form $F(x, y)\tau^+$.

Utilizing this circumstance, the expression (3) may be expressed as

$$\left.\frac{\partial G(x, y; I_\mu)}{\partial I_\mu}\right|_{I_\mu = 0} = i(x_\mu - y_\mu)\langle 0 \,|\, T\{\psi(x), \bar{\psi}(y)\} \,|\, 0\rangle = i(x_\mu - y_\mu)G(x, y)\tau^+ ,$$

and, therefore

$$(4)\qquad \Gamma^+_\mu(x, y; \xi) = -\, i(x_\mu - y_\mu)\tau^+ G^{-1}(y - x)\, \delta(\xi) .$$

In the momentum representation we obtain a correlation analogous to Ward's theorem in electrodynamics:

$$(5)\qquad \Gamma^+_\mu(p, p; 0) = \tau^+ \frac{\partial G^{-1}(p)}{\partial p_\mu} ;$$

The further proof of the absence of renormalization of charge is the same as in quantum electrodynamics, taking no account of vacuum polarization.

If besides $\pi$-meson-nucleon interactions the interactions of nucleons with K-mesons and hyperons are also considered, then for the absence of renormalization of the constant of $\beta$-interaction the group of transformations (2) must be extended to strange particles. That may be attained if the wave fuctions of K-mesons and $\Xi$-hyperons will be transformed in the same way as the wave functions of nucleons, the wave function of $\Sigma$-hyperons, as the wave functions of $\pi$-mesons, while the wave function of the $\Lambda^0$-particle will remain constant. The presence of such a group of transformation of strange particle wave functions, therefore, demands that the vector part of the Hamiltonian $\beta$-interaction of K-mesons and hyperons should have the following expression:

$$(6) \quad H = G_v[2\bar{\psi}_\Sigma \gamma_\mu T^+ \psi_\Sigma + i(\varphi_K^+ \tau^+ \nabla_\mu \varphi_K - (\nabla_\mu \varphi_K^+)\tau^+ \varphi_K) + \bar{\psi}_\Xi \gamma_\mu \tau^+ \psi_\Xi] I_\mu + \text{c. c.}$$

Hamiltonian (6) describes processes of $\beta$-decay of strange particles, which occur without change in strangeness (*) (such as $\Sigma^- \to \Sigma^0 + e + \nu$, $K^- \to K^0 + e + \nu$). The constant $G_v$ in the Hamiltonian (6) coincides with the constant in the Hamiltonian (1) and in the same manner as the latter is not renormalized by strong interactions.

It must be noted that in the proof given no account was taken of radiation corrections due to the electromagnetic field. Taking into account the interaction of particles with an electromagnetic field disturbs the group of transformations (2) and should, generally speaking, bring forth the renormalization of the constant $G_v$.

(*) Processes occurring with the variation of strangeness, seemingly, would not affect the renormalization of the value of the constant $G_v$ in processes, in which the strangeness does not change.


## RIASSUNTO (*)

Si dimostra che la costante d'accoppiamento vettoriale non è soggetta a rinormalizzazione se, secondo Feynman e Gell-Mann, si prende l'hamiltoniana del decadimento $\beta$ nella forma delle (1) e (6).


(*) *Traduzione a cura della Redazione.*

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# On Corollaries from the Two-Component Nature of the Electron in β-Interactions.

B. L. Ioffe and V. A. Ljubimov

*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — A necessary and sufficient condition for the value $v/c$ of the longitudinal polarization of the electron is discussed.

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The latest measurements [1-3] of the electron longitudinal polarization in β-decay demonstrate that the value of the longitudinal polarization $\langle\sigma_{\parallel}\rangle$ in the case of allowed transitions and first forbidden transitions in heavy nuclei equals $v/c$ with a good degree of accuracy. As it can be strictly shown from the formulae (4) for the electron longitudinal polarization in such transitions, the necessary and sufficient conditions for $\langle\sigma_{\parallel}\rangle = v/c$ is the existence of the following relationships between the parity conservation and non-conservation coupling constants (the designations employed hereafter are those adopted in [4]):

$$C_S = -C'_S, \quad C_T = -C'_T, \quad C_A = C'_A, \quad C_V = C'_V. \tag{1}$$

The Hamiltonian interactions then turn into

$$H = \sum_{\alpha} C_\alpha (\bar{\psi}_p O_\alpha \psi_n)(\bar{\psi}_e (1 - \gamma_5) O_\alpha \psi_\nu) + \text{c. c.} \tag{2}$$

and the electron $\psi$-function enters in all the variants of β-interaction in two components only.

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(1) A. I. Alihanov, G. P. Eliseev, V. A. Ljubimov: in print.
(2) P. Cavanagh C. Coleman, J. Turner and B. Ridler: in print.
(3) F. Boehm, T. Novey, C. Barnes an B. Steach: in print.
(4) V. B. Berestetckij, B. L. Ioffe, A. P. Rudik, K. A. ter Martirosjan: in print.

Let us consider certain corollaries from the relationship (1), *i.e.* from the two-component nature of the electron in β-interactions. Provided conditions (1) are fulfilled, the expressions for different effects in β-decay are substantially simplified so that in the case of allowed transitions there remain six independent combinations of constants and matrix elements only,

$$N_0 = (|C_S|^2 + |C_V|^2)|M_F|^2 + (|C_T|^2 + |C_A|^2)|M_{GT}|^2,$$

$$N_1 = -\lambda_{jj'}(|C_T|^2 + |C_A|^2)|M_{GT}|^2 - 2\delta_{jj'}\sqrt{j/(j+1)}\ \mathrm{Re}\,(C_S C_T^* + C_V C_A^*) M_F \cdot M_{GT}^*$$

$$N_3 = (|C_V|^2 - |C_S|^2)|M_F|^2 + \tfrac{1}{3}(|C_T|^2 - |C_A|^2)|M_{GT}|^2,$$

$$N_4 = 2\delta_{jj'}\sqrt{j/(j+1)}\ \mathrm{Im}\,(C_V C_A^* - C_S C_T^*) M_F \cdot M_{GT}^*,$$

$$N_\nu = -\lambda_{jj'}(|C_T|^2 - |C_A|^2)|M_{GT}|^2 + 2\delta_{jj'}\sqrt{j/(j+1)}\ \mathrm{Re}\,(C_S C_T^* - C_V C_A^*) M_F \cdot M_{GT}^*,$$

$$N_5 = 2\delta_{jj'}\sqrt{j/(j+1)}\ \mathrm{Im}\,(C_V C_A^* + C_S C_T^*) M_F \cdot M_{GT}^*.$$

Here $\lambda_{jj'} = [j(j+1) - j'(j'+1) + 2]/2(j+1)$ and $M_F = (\int 1)$, $M_{GT} = (\int \sigma)$ are the nuclear matrix elements. The quantity $N_0$ determines the full probability of the β-transition; $N_3$ determines the angular correlation of electron-neutrino

$$W_{e\nu} = 1 + (v/c)(N_3/N_0)(n_e \nu)$$

and $N_1$ is the angular distribution of electrons for an orientated nucleus

$$W_{je} = 1 + \chi(v/c)(N_1/N_0)(n_e n_j)$$

($n_e$, $\nu$, $n_j$ are the single direction vectors of electron and neutrino momenta and nuclear spin, $\chi = \langle j_z \rangle / j$). The quantities $N_0$, $N_1$ and $N_3$ were determined experimentally. As is evident from (3), fundamentally new information can now be obtained only from experiments involving measurements of $N_4$, $N_\nu$, and $N_5$. The quantity $N_\nu$ determines the neutrino angular distribution for an oriented nucleus (averages for the directions of electrons' emergence)

$$W_{j\nu} = 1 + \chi(N_\nu/N_0)(\boldsymbol{n}_j \boldsymbol{\nu}).$$

Therefore the simplest experiment in which the quantity $N_\nu$ can be determined is to measure the angular distribution of recoil nuclei in the decay of oriented nuclei (*). Coefficient $N_4$ can be measured by investigating the distribution asymmetry of recoil nuclei with respect to the plane passing through electron momentum and nuclear spin directions. Thus, if the electrons with

(*) An equivalent experiment is that dealing with the β-γ transition in measuring the correlation of the emergence directions of the γ-quantum of a given circular polarization and recoil nucleus.

momentum perpendicular to the nuclear spin are treated as fixed, the ratio of recoil nuclei whose direction of movement is on opposite sides of this plane is equal to

$$\left(1-\frac{\chi}{2}\frac{v}{c}\frac{N_4}{N_0}\right)\Big/\left(1+\frac{\chi}{2}\frac{v}{c}\frac{N_4}{N_0}\right),$$

The quantity $N_5$ can be obtained from experiments on the decay of oriented (or aligned) nuclei, in which is mesured the direction of the emerged electron and the polarization of the recoil nucleus or the $\gamma$-quantum direction in subsequent $\gamma$ transitions (*).

Measurements of the electron polarization (both longitudinal and transverse) for orientated nuclei and in correlation with neutrino can yield nothing new as compared with the experiments described above. Thus, the electron polarization in the decay of an orientated nucleus equals

$$\langle\boldsymbol{\sigma}\rangle_{je}=\frac{1}{W_{je}}\left\{-\boldsymbol{n}_e\frac{v}{c}\left[1+\chi\frac{c}{v}\frac{N_1}{N_0}(\boldsymbol{n}_e\boldsymbol{n}_j)\right]+\chi\frac{z}{137\varepsilon}\bar{\eta}_0\frac{N_1}{N_0}[\boldsymbol{n}_e\boldsymbol{n}_j]+\chi\frac{\gamma_1\bar{\mu}_0}{\varepsilon}\frac{N_1}{N_0}[\boldsymbol{n}_e[\boldsymbol{n}_e\boldsymbol{n}_j]]\right\},$$

and the correlation of the polarization with the direction of neutrino emergence (for non-polarized nucleus) equals

$$\langle\boldsymbol{\sigma}\rangle_{e\nu}=\frac{1}{W_{e\nu}}\left\{-\boldsymbol{n}_e\frac{v}{c}\left[1+\frac{c}{v}\frac{N_1}{N_0}(\boldsymbol{n}_e\boldsymbol{\nu})\right]+\frac{z}{137\varepsilon}\bar{\eta}_0\frac{N_3}{N_0}[\boldsymbol{n}_e\boldsymbol{\nu}]+\frac{\gamma_1\bar{\mu}_0}{\varepsilon}\frac{N_3}{N_0}[\boldsymbol{n}_e[\boldsymbol{n}_e\boldsymbol{\nu}]]\right\}.$$

Here $\varepsilon$ is the electron energy (in $m_ec^2$ units), $\gamma_1=\sqrt{1-(z/137)^2}$, $\bar{\mu}_0$ and $\bar{\eta}_0$ are efficients close to unity, taking into account the finite dimensions of the nucleus. Thus, the measurements of $\langle\boldsymbol{\sigma}\rangle_{je}$ and $\langle\boldsymbol{\sigma}\rangle_{e\nu}$ do not yield, in principle, any new information as compared with earlier experiments in which the quantities $N_0$, $N_1$, and $N_3$ have been measured.

It is easy to see that when the coefficients $C_\alpha$ are written in terms of $|C_\alpha|\exp[i\varphi\alpha]$, the quantities $N_0$, $N_1$, $N_3$, $N_4$, $N_\nu$, $N_5$ will be expressed in 6 unknown coefficients only: four moduli $|C_\alpha|$ and two phase differences $\varphi_T-\varphi_S$ and $\varphi_\nu-\varphi_A$ since, in the case of a two-component electron, the $A$ and $V$ variants do not interfere with $S$ and $T$. Therefore to obtain full information about the $\beta$-interaction in allowed transitions there is in principle no need to measure

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(*) The determination of the quantity $N_5$ from the experiments on $\beta$-$\gamma$ correlation in the decay of oriented (or aligned nuclei) was treated in detail in [5-7].

([5]) A. Z. Dolginov: in print.
([6]) M. Morita and R. S. Morita: *Phys. Rev.*, **107**, 1316 (1957).
([7]) R. B. Curtis and R. P. Lewis: *Phys. Rev.*, **107**, 1381 (1957).

experimentally all the six values of $N_i$ in Fermi- and Gamov-Teller transitions. It is sufficient to consider four of them only.

* * *

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## RIASSUNTO (*)

Si discute una condizione necessaria e sufficiente per il valore $v/c$ della polarizzazione longitudinale dell'elettrone.

(*) *Traduzione a cura della Redazione.*

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# On the Polarization of Electrons from the Decay of Muons.

L. B. Okun and V. M. Šehter
*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — An expression for the energy spectrum, angular distribution and polarization of electrons emitted by decaying muons has been obtained (formula (2) in the Appendix). The calculations are carried out for a decay interaction Hamiltonian of the most general form characterized by ten complex constants (formula (1) in the Appendix). Results, obtained previously by different authors ([9-21]), are contained in formula (2). It is shown that experiments performed up to the present time are not sufficient for the verification of the predictions made in references ([4,5]). According to these references, conditions (2) and (3) must be fullfilled; the distribution of electrons must be described by the formula (4). It follows from (2) that a combination of $S$, $T$ and $P$ couplings gives formula (5), which differs only by the sign from (4). Hence to decide which of these two formulas is the correct one, the polarization sign of the decaying muons must be measured.

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We have obtained an expression for the energy spectrum, the angular distribution and the polarization of the electrons from the decay of polarized $\mu$-mesons. Calculations are made with the most general Hamiltonian of the decay interaction, characterized by ten complex constants. We have demonstrated that the experiments already made are insufficient to verify the validity of the assumptions which have been made in the works ([4]) and ([5]); to complete this experiment it is necessary to change the sign of the polarization of the $\mu$-mesons which decay.

Experimental data of the energy spectrum, asymmetry, and polarization of electrons in the decay of polarized $\mu$-mesons apparently agree with the hypothesis of the theory of the two-component neutrino, which has been

assumed by SALAM [1], LANDAU [2] and LEE and YANG [3]. In the light of this theory — if we take into account the experimentally observable aspect of the electrons — of the ten complex constants $C$ and $C'$ which describe in the general case the decay of the $\mu$-mesons (cfr. formula (1) of Appendix), only four are different from 0.

$$C_V = C'_A \neq 0\,, \qquad C_A = C'_V \neq 0\,, \tag{1}$$

$$C_S = C'_S = C_P = C'_P = C_T = C'_T = 0\,. \tag{2}$$

One can hope that more precise experimental data will agree with the more stringent conditions of FEYNMAN and GELL-MANN [4] and of MARSHAK and SUDARSHAN [5] for an interaction with an electron with two components, for which

$$C_V = \pm C'_V\,, \qquad C_A = \pm C'_A\,. \tag{3}$$

In this last case, the distribution of electrons in the decay of $\mu$-mesons must be proportional to

$$(1 \mp \boldsymbol{\zeta}\cdot\boldsymbol{\eta})[3 - 2\varepsilon \pm \boldsymbol{\eta}\cdot\boldsymbol{n}(1 - 2\varepsilon)]\varepsilon^2\,\mathrm{d}\varepsilon\,. \tag{4}$$

Here: $\varepsilon$ is the energy of the electron divided by its maximum value,
$\boldsymbol{n}$ is a unit vector in the direction of motion of the electron,
$\boldsymbol{\zeta}$ is a unit vector in the direction of the spin of the electron,
$\boldsymbol{\eta}$ is a unit vector in the direction of the spin of the $\mu$-meson in the rest system of the $\mu$-meson.

One can obtain formula (4) from the formulae (2) of the Appendix if the conditions (1), (2) and (3) are satisfied and if one neglects the mass of the electron $M_e$ as compared to its energy.

The first term in the formula (4) indicates that the electron must be completely polarized longitudinally. It is experimentally established [6,7] that the positrons are polarized in the direction of motion in the decay of $\mu^+$-mesons. Therefore the lower signs in formula (4) refer to the decay of $\mu^+$-mesons, and the upper ones to the decay of $\mu^-$-mesons. Furthermore, as the positrons with maximum energy move in the direction opposite to the impulse of the

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(1) A. SALAM: *Nuovo Cimento*, **5**, 299 (1957).
(2) L. LANDAU: *Žu. Éksper. Teor. Fiz.*, **32**, 405 (1957).
(3) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957).
(4) R. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1957).
(5) R. MARSHAK and E. SUDARSHAN: *Phys. Rev.* (in the press).
(6) M. LEDERMANN: *Rochester Conference* (1957).
(7) C. CULLIGAN, S. C. F. FRANK and J. C. KLYVER: *Venice Conference* (1957).

$\mu^+$-meson ([8]) and as following formula (4), they move in the direction of the spin of the $\mu^+$-meson, then if formula (4) is correct one can conclude that the spin of the $\mu^+$-meson which arises from the $\pi^+$-meson decay is directed opposite to the direction of the $\mu^+$-meson momentum.

It would be very interesting to obtain an experimental verification of this fact. It would also be interesting to clarify to what degree the experimental verification of formula (4) will be significant for the verification of the theory of the two-component neutrino. In other words, if the electrons suffer an interaction with only two components, that is $C_i = \pm C_i'$, the question then is whether both conditions (1) and (2) in this case are not only sufficient for the verification of formula (4), but whether they are also necessary.

As concerns condition (1), it is obvious that if, let us suppose for example, $C_A = C_A' = 0$ and $C_V = \pm C_V' \neq 0$, then formula (4) will be verified. It is easy to verify this result with the aid of formulae (2) of the appendix, in such a manner that condition (1) is not necessary for the validity of formula (4), and an experimental verification of this formula will not indicate that the neutrino has only two components.

As concerns condition (2): it follows from the results of a series of works ([9-12]) that one can obtain the same spectrum and the same asymmetry of electrons as in formula (4) if one supposes that $C_V = C_V' = C_A = C_A' = 0$ and correspondingly one uses the constants $C_S$, $C_S'$, $C_P$, $C_P'$, $C_T$, $C_T'$. It is clear that if one is not interested in the asymmetry of the electron, then it is easy to obtain the necessary sign of its polarization, the sign being observable by experiment. Then the question is: can one, using a suitable combination of $S$, $P$ and $T$ terms, get a formula which coincides with formula (4)?

The analysis of formula (2) gives a negative answer to this question. Actually, by combination of $S$, $P$ and $T$ terms one can obtain

$$(1 \mp \zeta\eta)[3 - 2\varepsilon \mp \eta n(1 - 2\varepsilon)]\varepsilon^2 \, d\varepsilon \,, \tag{5}$$

which differs from (4) by the sign of the asymmetry of the electron if the sign of its polarization is given. Therefore, if it is demonstrated that the polarization of the $\mu^+$-meson in the decay of $\pi^+$-meson agrees with the conclusions of the two-component theory of the neutrino, which we have considered above, then this will indicate that formula (5) is not correct. Therefore, the condition (2) is necessary for the verification of formula (4).

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([8]) R. L. Garwin, L. M. Ledermann and M. Weinrich: *Phys. Rev.*, **105**, 1415 (1957).
([9]) L. B. Okun and R. Rudik: *Žu. Éksper. Teor. Fiz.*, **32**, 627 (1957).
([10]) C. Buchiat and L. Michel: *Phys. Rev.*, **106**, 170 (1957).
([11]) T. Kinoshita and A. Sirlin: *Phys. Rev.*, **107**, 593 (1957).
([12]) S. Larsen, E. Lubkin and M. Tausner: *Phys. Rev.*, **107**, 853 (1957).

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We have investigated the necessity of conditions (1) and (2), supposing that the condition of two-component electron interaction is satisfied. If one finds that this last condition is not verified by experiment, then it is easy to make some analogous considerations with the aid of formula (2) of the Appendix even in this more general case.

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The authors extend their thanks to V. N. GUMEN for his aid in the calculations.

## APPENDIX

Here we consider the result of the calculation of the asymmetry and polarization of the electrons in the decay of the polarized $\mu$-meson. Calculation is made for the case where the Hamiltonian is the following

$$(1)\qquad \begin{cases} H = \sum\limits_{i=S,V,T,A,P} (\bar{\psi}_e(C_i + C_i'\gamma_5)O_i\psi_\mu(\bar{\psi}_\nu(O_i\psi_\nu)\,, \\ O_S = 1\,, \quad O_V = -i\gamma_\alpha,\ O_T = \dfrac{i}{2\sqrt{2}}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)\,, \\ O_A = \gamma_\alpha\gamma_5\,, \quad O_P = \gamma_5\,, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,. \end{cases}$$

It is convenient to make calculations using the method of Lenard ([13]) and the spinor projection operator of Michel and Wightman ([14]). The probability that in the decay of a $\mu$-meson with a spin parallel to $\boldsymbol{\eta}$, an electron emerges in the direction $\boldsymbol{n}$, with an energy $\varepsilon$ and with a spin in the direction $\boldsymbol{\zeta}$, is given by the following formula:

$$(2)\qquad \begin{cases} \mathrm{d}^2W/\mathrm{d}\varepsilon\,\mathrm{d}\Omega = (\mu\omega^4/96\pi^4)\sqrt{\varepsilon^2 - u^2}\{3S + 2V + 2T\}, \\ S = (C_SC_S^* + C_P'C_P'^*)(1-\varepsilon)(\varepsilon+u)\{1 + [\boldsymbol{\eta\zeta} - (\boldsymbol{\eta n})(\boldsymbol{\zeta n})] + (\eta n)(\zeta n)\} + \\ \quad + (C_S'C_S'^* + C_PC_P^*)(1-\varepsilon)(\varepsilon-u)\{1 - [\boldsymbol{\eta\zeta} - (\boldsymbol{\eta n})(\boldsymbol{\zeta n})] + (\eta n)(\zeta n)\} + \\ \quad + (C_SC_S'^* + C_S'C_S^* + C_PC_P'^* + C_P'C_P^*)(1-\varepsilon\sqrt{\varepsilon^2 - u^2}\{\boldsymbol{\eta n} + \boldsymbol{\zeta n}\} + \\ \quad + i(C_S'C_S^* - C_SC_S'^* + C_PC_P'^* - C_P'C_P^*)(1-\varepsilon)\sqrt{\varepsilon^2 - u^2}\,\boldsymbol{\zeta}[\boldsymbol{\eta n}]\,, \end{cases}$$

([13]) A. LENARD: *Phys. Rev.*, **90**, 968 (1953).
([14]) L. MICHEL and A. S. WIGHTMAN: *Phys. Rev.*, **98**, 1190 (1955).

$$
\begin{cases}
V = (C_V C_V^* + C'_A C'^*_A)(\varepsilon - u)\{(3 - 2\varepsilon + u) + (1 + u)[\eta\zeta - (\eta n)(\zeta n)] - \\
\quad - (1 - 2\varepsilon - u)(\eta n)(\zeta n)\} + (C'_V C'^*_V + C_A C_A^*)(\varepsilon + u)\{(3 - 2\varepsilon - u) - \\
\quad - (1 - u)[\eta\zeta - (\eta n)(\zeta n)] - (1 - 2\varepsilon + u)(\eta n)(\zeta n)\} + \\
\quad + (C_V C'^*_V + C'_V C_V^* + C_A C'^*_A + C'_A C_A^*)\sqrt{\varepsilon^2 - u^2}\{(1 - 2\varepsilon + v)\eta n - \\
\quad - (3 - 2\varepsilon - v)\zeta n\} + i(C'_V C_V^* - C_V C'^*_V - C'^*_A - C'_A C_A^*)\sqrt{\varepsilon^2 - u^2}(1 - v)\zeta[\eta n]\,. \\
T = (C_T C_T^* + C'_T C'^*_T)\{\varepsilon(3 - \varepsilon) - 2u^2 - u(1 - \varepsilon)[\eta\zeta - (\eta n)(\zeta n)] - \\
\quad - [\varepsilon(1 + \varepsilon) - 2u^2](\eta n)(\zeta n)\} + (C_T C'^*_T + C'_T C_T^*)\sqrt{\varepsilon^2 - u^2}\cdot \\
\quad \cdot\{(3 - \varepsilon - 2v)n\zeta - (1 + \varepsilon - 2v)n\eta\}; \\
\varepsilon = E_e/w\,, \quad u = m_e/w\,, \quad v = m_\nu^2/m_\mu w\,, \quad w = (m_\mu^2 + m_e^2)/2m_\mu\,.
\end{cases}
\tag{2}
$$

An expression proportional to $\zeta[n\eta]$ determines the transversal polarization of the electrons perpendicular to the plane $\eta n$. In the case of $T$ conservation when all constants $C$ and $C'$ are real, such a polarization, as we can see from formula (2), vanishes. An expression proportional to $[\eta\zeta - (\eta n)(\zeta n)]$ determines the transversal polarization of the electron in the plane $\eta n$.

If one makes the condition that formula (2) describes the decay of $\mu^+$-mesons, then to obtain a formula for the decay of $\mu^-$-mesons one has to make the following changes:

$$
\begin{cases}
C_S, C_A, C_P, C'_V, C_T \to \quad C_S^*, \quad C_A^*, \quad C_P^*, \quad C'^*_V, \quad C'^*_T\,, \\
C'_S, C'_A, C'_P, C_V, C_T \to -C'^*_S, -C'^*_A, -C'^*_P, -C_V^*, -C_T^*\,.
\end{cases}
\tag{3}
$$

Some partial results of the formula (2) were already obtained in a series of works. Thus if one averages formula (2) with respect to spin states of the $\mu$-meson and of the electron ($\eta = 0$, $\zeta = 0$), then it will reduce to a formula for the spectrum of the electrons, which was obtained for the first time by TIOMNO, WHEELER and RAU [15], and by MICHEL [13]. If one averages formula (2) only with respect to the polarization of the electron ($\zeta = 0$), then it will reduce to an expression which was obtained by RUDIK and one of the authors [9] for the case of $M_e = 0$, by BUCHIAT and MICHEL [10] for the case of $M_e \neq 0$, by KINOSHITA and SIRLIN [11] in the case $M_e = 0$, by LARSEN, LUBKIN and TAUSNER [12] in the case $M_e = 0$. If one supposes that conditions (1) and (2) are satisfied, then one gets from the formula (2) a formula which is analogous to that of KINOSHITA and SIRLIN [1] for the case $M_e = 0$ and to that of ÜBERALL [1], and CANDLIN [1] for $M_e \neq 0$. A formula anal-

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[15] J. TIOMNO, J. A. WHEELER and R. R. RAU: *Rev. Mod. Phys.*, **21**, 144 (1949).
[16] L. MICHEL: *Proc. Phys. Soc.*, A **63**, 514 (1950).
[17] T. KINOSHITA and A. SIRLIN: *Phys. Rev.*, **107**, 1110 (1957).
[18] H. ÜBERALL: *Nuovo Cimento*, **6**, 376 (1957).
[19] D. J. CANDLIN: *Nuovo Cimento*, **6**, 390 (1957).

ogous to the formula (2) for the case where only tensor interaction is absent, was obtained by SHARP and BACH [20], ($M_e \neq 0$). If one sets all primed constants $= 0$ and one puts $M_e = 0$, then formula (2) goes into the formula which was obtained by one of the authors [21] for the parity conserving Hamiltonian.

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[20] R. T. SHARP and G. BACH: *Can. Journ. Phys.*, **35**, 1199 (1957).
[21] L. B. OKUN: *Dokl. Akad. Nauk SSSR*, **104**, 840 (1955).

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## RIASSUNTO (*)

Si è ottenuta (formula (2) dell'appendice) un'espressione per lo spettro d'energia, la distribuzione angolare e la polarizzazione degli elettroni emessi nel decadimento dei muoni. Si sono eseguiti i calcoli per un'Hamiltoniana dell'interazione di decadimento di forma generica caratterizzata da dieci costanti complesse (formula (1) dell'Appendice). I risultati ottenuti precedentemente da differenti autori [9-21] sono contenuti nella formula (2). Si dimostra che le esperienze eseguite finora non sono sufficienti per la verifica delle previsioni fatte nei lavori [4,5]. Secondo questi lavori, debbono verificarsi le condizioni (2) e (3); la distribuzione degli elettroni deve essere descritta dà la (4). Segue da (2) che una combinazione di accoppiamenti $S$, $T$ e $P$ dà la (5), che differisce dalla (4) solo per il segno. Pertanto, per decidere quale delle due espressioni è corretta si deve misurare la polarizzazione dei muoni che decadono.

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# Polarization of Electrons of Integral Conversion Following the β-Decay Taking into Account the Electric Field of the Nucleus.

B. V. Geškenbein

*Academy of Sciences - Moscow*

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**Summary.** — The correlation of the polarization of conversion electrons with the direction of emission electrons of the preceding β-decay has been considered. In the entire computation the electric field of the nucleus has been taken into account. Computation of the electric field of the nucleus results in the appearance in magnetic transitions of a considerable transversal component of polarization, while in the electrical transitions the magnitude of polarization is increased substantially.

## 1. – Introduction.

In view of the fact that parity is not conserved in β-decay, the nucleus resulting after β-decay, will be polarized in the direction of the emitted β-electron. The decomposing nucleus is assumed to be non-polarized, and the direction of emission of the neutrino is not registered. Therefore, if after the β-decay a process of internal conversion takes place, the conversion electrons should possess a definite polarization. This phenomenon, without taking into account the electric field of the nucleus, was considered by V. B. Berestetsky and A. P. Rudik in [1]. The internal conversion is substantially affected by the value of the nucleus' field, which greatly changes the coefficient of conversion. Therefore, the electric field of the nucleus should influence substantially also the polarization of the conversion electrons.

The influence of the electric field of the nucleus shows itself in the fact that the wave function of the emitted electron is not a plane wave, but is

(1) V. B. Beresteckij and A. P. Rudnik: *Žu. Eksper. Teor. Fiz.* (in print).

such a solution of the equation of Dirac, which at infinity is a superposition of a plane and converging spherical waves. This function was constructed in [2] and we shall write it down in the form:

$$\psi_2 = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{p_2 \varepsilon_2}} \sum_{j_2 l_2 \mu_2} [\Omega^{(l_2)}_{j_2 \mu_2}(\boldsymbol{n})]^*_\xi \exp[-i\delta_{\varkappa_2}] \begin{pmatrix} i g_{\varkappa_2}(r) \Omega^{(l_2)}_{j_2 \mu_2}\left(\frac{\boldsymbol{r}}{r}\right) \\ f_{\varkappa_2}(r)\, \Omega^{(l_2')}_{j_2 \mu_2}\left(\frac{\boldsymbol{r}}{r}\right) \end{pmatrix}, \tag{1}$$

where

$$[\Omega^{(l_2)}_{j_2 \mu_2}(\boldsymbol{n})]_\xi = C^{j_2 \mu_2}_{\frac{1}{2}\xi;\, l_2, \mu_2 - \xi} y_{l_2, \mu_2 - \xi}(\boldsymbol{n}),$$

$\xi$ is a projection of a two-component spherical spinor;
$\boldsymbol{n}$ the unit vector in the direction of the emitted electron;
$C^{c\gamma}_{a\boldsymbol{\alpha};\, b\beta}$ the Clebsch-Gordan coefficients;

$$l_2' = 2j_2 - l_2; \quad \varkappa_2 = (l_2 - l_2')(j_2 + \tfrac{1}{2}).$$

$g_{\varkappa_2}(r)$ and $f_{\varkappa_2}(r)$ are radial parts of the Coulomb functions, which are normalized for the $\delta$-function from energy

$$\delta_{\varkappa_2} = \text{phase at } Z \to 0, \qquad \delta_{\varkappa_2} \to -\frac{\pi}{2}(l_2 - 1).$$

The wave function of an electron on the $K$-shell has the form

$$\psi_1 = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} i g(r) u_{\mu_1} \\ f(r) \boldsymbol{\sigma} \frac{\boldsymbol{r}}{r} u_{\mu_1} \end{pmatrix}. \tag{2}$$

Here and everywhere further on we shall use the system of units, in which $\hbar = c = 1$.

The matrix element of the process of conversion may be written down in the following form (3) (the insignificant common denominators are dropped)

$$\mathcal{M}_{m_1 m_2} = (J_1 m_1 | Q^{(\lambda)}_{j\mu} | J_2 m_2)^* (B^{(\lambda)}_{j\mu})_{21}, \tag{3}$$

$$(B^{(\lambda)}_{j\mu})_{21} = \int \psi_2^* B^{(\lambda)}_{j\mu} \psi_1 \, \mathrm{d}v; \tag{3a}$$

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[2] V. B. Berestetckij, B. L. Ioffe, A. P. Rudnik and K. A. Ter-Martirosjan: *Nuclear Physics*, **5**, 464 (1958).

$(J_1 m_1 | Q^{(\lambda)}_{j\mu} | J_2 m_2)$ = nuclear transition matrix element,

$J_1, m_1$ spin and projection of nucleus spin before conversion,

$J_2, m_2$ spin and projection of nucleus spin after conversion,

$B^{(\lambda)}_{j\mu}$ the operator of interaction of an electron with the field of the multipole.

This operator has the following form:

$$B^{(\lambda)}_{j\mu} = G_j(wr)\boldsymbol{\alpha}\mathbf{y}_{jj\mu}\left(\frac{\boldsymbol{r}}{r}\right), \tag{4}$$

$$B^{(1)}_{j\mu} = G_j(wr)y_{j\mu}\left(\frac{\boldsymbol{r}}{r}\right) + \sqrt{\frac{2j+1}{j}}\, G_{j-1}(wr)\boldsymbol{\alpha}\mathbf{y}_{jj-1,\mu}\left(\frac{\boldsymbol{r}}{r}\right). \tag{5}$$

Value $\lambda = 0$ corresponds to the transition of magnetic type, while $\lambda = 1$ to the transition of electric type,

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \boldsymbol{\sigma} = \text{Pauli matrices}, \quad w = \text{energy of the transition},$$

$$G_j(x) = (2\overline{n})^{\frac{1}{2}}\, i^j \frac{H^{(1)}_{j+\frac{1}{2}}(x)}{\sqrt{x}} \qquad (H^{(1)} \text{ a function of Hankel}),$$

$y_{j\mu}$ — spherical function, spherical vector with contravariant components $(\mathbf{y}_{jl\mu})^\nu = C^{j\mu}_{l,\,\mu-\nu;\,1\nu} y_{l,\,\mu-\nu}$.

In relation to the spin states of a conversion electron the density matrix has the form:

$$P = \varrho_{m_1 m_1'}(J_1 m_1 | Q^{(\lambda)}_{j\mu} | J_2 m_2)^* (J_1 m_1' | Q^{(\lambda)}_{j\mu'} | J_2 m_2)(B^{(\lambda)}_{j\mu})_{21}(B^{(\lambda)}_{j\mu'})^*_{21}. \tag{6}$$

The expression for the density matrix $\varrho_{m_1 m_1'}$ characterizing the polarized state of the nucleus after β-decay was obtained in [1] and has the form:

$$\varrho_{m_1 m_1'} = \left(\delta_{m_1 m_1'} + \sqrt{\frac{J_1+1}{J_1}}\, \alpha C^{J_1 m_1}_{J_1 m_1';\,1\gamma} V^\gamma\right) \frac{1}{2J_1+1}. \tag{7}$$

Here $V^\nu$ spherical components of velocity of a β-electron

$$\left(V^{(0)} = V_z,\ V^{\pm 1} = \mp\frac{1}{\sqrt{2}}(V_x \mp V_y)\right);$$

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[3] A. U. Ahiezer and B. V. Berestetckij: *Quantum Electrodynamics.*

$\alpha$ a constant, determining the angular distribution of the β-electrons from the polarized nuclei:

$$W = 1 + \alpha \frac{\boldsymbol{V}\langle \boldsymbol{J}_1 \rangle}{J_1}. \tag{8}$$

For the allowed transitions in case of $S$, $T$, $A$, $V$ variants $\alpha$ is equal [4] to $\alpha = C_1 + C_2(zc^2/\varepsilon v)$

$$C_1\zeta = 2\,\mathrm{Re}\left\{(C_T C_T'^* - C_A C_A'^*)\,|M_{GT}|^2 \lambda_{J_1 J} + \right.$$
$$\left. + \sqrt{\frac{J_1}{J_1+1}}\,(C_T C_S'^* + C_T' C_S^* - C_A C_V'^* - C_A' C_V^*) M_F M_{GT}^*\right\},$$

$$C_2\zeta = 2\,\mathrm{Jm}\left\{(C_T C_A'^* + C_T' C_A^*)\,|M_{GT}|^2 \lambda_{J J} + \right.$$
$$\left. + \sqrt{\frac{J_1}{J_1+1}}\,(C_T C_V'^* + C_T' C_V^* - C_A C_S'^* - C_A' C_S^*) M_F M_{GT}^*\right\},$$

$$\zeta = (|C_T|^2 + |C_T'|^2 + |C_A|^2 + |C_A'|^2)\,|M_{GT}|^2 + (|C_S|^2 + |C_S'|^2 + |C_V|^2 + |C_V'|^2)\,|M_F|^2,$$

$$\lambda_{J J} = \begin{cases} \dfrac{1}{J+1} & J_1 = J \\ 1 & J_1 = J+1 \\ -\dfrac{J-1}{J} & J_1 = J-1 \end{cases} \tag{9}$$

Let us note that the matrix $\varrho_{m_1 m_1'}$ may be written in the form of (7) only for the allowed transitions and in the Coulomb transitions of the first forbiddance.

## 2. – Magnetic transition.

Substituting in the expression (3$a$) wave function of an electron $\psi_1$ and $\psi_2$ from (1) and (2) and the operator $B_{j\mu}^{(0)}$, we get

$$(B_{j\mu}^{(\gamma)})_{21} = \sum_{j_2 l_2 \mu_2} [\Omega_{j_2 \mu_2}^{(j_2)}(\boldsymbol{n})]_\xi \exp[i\delta_{\varkappa_2}] \left\{ R_{1\varkappa_2} \int \Omega_{j_2 \mu_2}^{(l_2')}\left(\frac{\boldsymbol{r}}{r}\right) \boldsymbol{\sigma} \boldsymbol{y}_{j j_2 \mu} u_{\mu_1} \mathrm{d}0 - \right.$$
$$\left. - R_{2\varkappa_2} \int \Omega_{j_2 \mu_2}^{(l_2)*}\left(\frac{\boldsymbol{r}}{r}\right) (\boldsymbol{\sigma} \boldsymbol{y}_{j j_2 \mu}) \left(\boldsymbol{\sigma} \frac{\boldsymbol{r}}{r}\right) u_{\mu_1} \mathrm{d}0 \right\}, \tag{10}$$

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[4] I. M. Šmouškovič: *Žu. Èksper. Teor. Fiz.*, **33**, 1477 (1957).

where

$$(11)\qquad \begin{cases} R_{1\varkappa_2} = \int f_{\varkappa_2}(r) g(r) G_j(wr) r^2 \, \mathrm{d}r \,, \\ R_{2\varkappa_2} = \int g_{\varkappa_2}(r) f(r) G_j(wr) r^2 \, dr \,. \end{cases}$$

For calculating angular integrals in (6) let us make use of the formula

$$\boldsymbol{y}_{jj\mu} = \frac{\boldsymbol{L} y_{j\mu}}{\sqrt{j(j+1)}} \,, \qquad (\boldsymbol{L} = -i[\boldsymbol{r}\nabla]) \,,$$

of the property of the self-conjugation operator $\boldsymbol{\sigma L}$ and also of the fact that

$$\boldsymbol{\sigma L}\Omega^{(l)}_{j\mu} = \left\{ j(j+1) - l(l+1) - \frac{3}{4} \right\} \Omega^{(l)}_{j\mu}; \quad \boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\,\Omega^{(l)}_{j\mu}\left(\frac{\boldsymbol{r}}{r}\right) = \Omega^{(l')}_{j\mu}\left(\frac{\boldsymbol{r}}{r}\right); \quad \boldsymbol{n}\boldsymbol{y}_{jj\mu}(\boldsymbol{n}) = 0 \,.$$

Then

$$(12)\quad \int \Omega^{(l')*}_{j_2\mu_2}\, \boldsymbol{\sigma}\boldsymbol{y}_{jj\mu} u_{\mu_1}\, \mathrm{d}0 = -\int \Omega^{(l_2)*}_{j_2\mu_2} (\boldsymbol{\sigma}\boldsymbol{y}_{jj\mu}) \boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\, u_{\mu_1}\, \mathrm{d}0 =$$

$$= \begin{cases} \sqrt{\dfrac{j}{j+1}}\, C^{j_2\mu_2}_{j\mu;\frac{1}{2}\mu_1}\, \delta_{l'_2 j} & j_2 = j + \tfrac{1}{2} \,, \\ \sqrt{\dfrac{j+1}{j}}\, C^{j_2\mu_2}_{j\mu;\frac{1}{2}\mu_1}\, \delta_{l'_2 j} & j_2 = j - \tfrac{1}{2} \,, \end{cases}$$

$$(13)\qquad \varkappa_2 = \begin{cases} j+1 & j_2 = j + \tfrac{1}{2} \,, \\ -j & j_2 = j - \tfrac{1}{2} \,. \end{cases}$$

After values of angular integrals (12) in the formula (6) are substituted, we get

$$(14)\quad (B^{(0)}_{j\mu})_{21} = \sqrt{\frac{j+1}{j}}\,(R_{1,-j} + R_{2,-j}) C^{j-\frac{1}{2},\mu+\mu_1}_{j\mu;\frac{1}{2}\mu_1} C^{j-\frac{1}{2},\mu+\mu_1}_{j-1,\mu+\mu_1,-\xi;\frac{1}{2}\xi} \exp[i\delta_{-j}] y_{j-1,\mu+\mu_1-\xi}(\boldsymbol{n}) -$$

$$- \sqrt{\frac{j}{j+1}}\,(R_{1,j+1} + R_{2,j+1}) C^{j+\frac{1}{2},\mu+\mu_1}_{j\mu;\frac{1}{2}\mu_1} C^{j+\frac{1}{2},\mu+\mu_1}_{j+1,\mu+\mu_1,-\xi} \exp[i\delta_{j+1}]\, y_{j+1,\mu+\mu_1-\xi}(\boldsymbol{n}) \,.$$

With precision up to common factors the expression (14) can be rewritten in a more convenient form:

$$(15)\qquad (B^{(0)}_{j\mu})_{21} = \boldsymbol{y}^{(1)}_{j\mu}\boldsymbol{\sigma} + A^{(0)}\boldsymbol{y}^{(-1)}_{j\mu}\boldsymbol{\sigma} \,,$$

where

$$\boldsymbol{y}^{(1)}_{j\mu} = \sqrt{\frac{j}{2j+1}}\,\boldsymbol{y}_{j,j+1,\mu} + \sqrt{\frac{j+1}{2j+1}}\,\boldsymbol{y}_{j,j-1,\mu}; \quad \boldsymbol{y}^{(-1)}_{j\mu} = \boldsymbol{n}y_{j\mu},$$

$$A^{(0)} = \frac{\sqrt{j(j+1)}\,\{R_{1,-j} + R_{2,-j} - (R_{1,j+1} + R_{2,j+1})\exp[i(\delta_{j+1} - \delta_{-j})]\}}{(j+1)(R_{1,-j} + R_{2,-j}) + j(R_{1,j+1} + R_{2,j+1})\exp[i(\delta_{j+1} - \delta_{-j})]}. \tag{16}$$

For $(B^{(0)}_{j\mu})_{21}(B^{(0)}_{j\mu'})^*_{21}$ we will get the expression

$$(B^{(0)}_{j\mu})_{21}(B^{(0)}_{j\mu'})^*_{21} = \boldsymbol{y}^{(0)}_{j\mu}\boldsymbol{y}^{(0)*}_{j\mu'} + |A^{(0)}|^2 y_{j\mu}y^{*}_{j\mu'} + \{i[\boldsymbol{y}^{(0)}_{j\mu}\boldsymbol{y}^{(0)*}_{j\mu'}] + A^{(0)}y_{j\mu}\boldsymbol{y}^{(0)*}_{j\mu'} + A^{(0)*}y^{*}_{j\mu'}\boldsymbol{y}^{(0)}_{j\mu}\}\boldsymbol{\sigma}, \tag{17}$$

where $\boldsymbol{y}^{(r)}_{j\mu} \equiv \boldsymbol{y}_{jj\mu}$.

Having made the same computations as in (1), we will get the following expression for the polarization of conversion electrons

$$\langle\boldsymbol{\sigma}\rangle = \alpha\frac{j(j+1) + J_1(J_1+1) - J_2(J_2+1)}{2j(j+1)J_2(1+|A^{(0)}|^2)} \cdot \{(\boldsymbol{Vn})\boldsymbol{n} + \sqrt{j(j+1)}\,\mathrm{Re}\,A^{(0)}(\boldsymbol{V} - (\boldsymbol{Vn})\boldsymbol{n})\}. \tag{18}$$

Let us consider two limiting cases:

1) $Z$ is small. The electric field of the nucleus may be neglected. In this case radial integrals are calculated in the obvious form:

$$R_{1,\varkappa_2} = (2\bar{n}Ze^2m)^{\frac{3}{2}}\sqrt{\frac{2(\varepsilon_2 - m)}{w}}\,I_j i^j\,\frac{\varkappa_2}{|\varkappa_2|},$$

$$R_{2,\varkappa_2} = 0; \qquad I_j = \int J_{j+\frac{1}{2}}(p_2 r)H^{(1)}_{j+\frac{1}{2}}(wr)r\,\mathrm{d}r = \frac{2}{\pi i}\left(\frac{p_2}{w}\right)^{j+\frac{1}{2}}\frac{1}{p_2^2 - w^2};$$

$$\varepsilon_2 = w + m; \qquad \delta_{-j} - \delta_{j+1} = \pi.$$

Therefore $A^{(0)} = 0$ and the expression for the polarization coincides with that given in (1).

2) $Z$ is great while the energy of the conversion electron is small, so that $p_2/mze^2 \ll 1$. In this case $R_{1,j+1} \ll R_{1,-j}$, $R_{2,j+1} \ll R_{2,-j}$.

Consequently $A^{(0)} \approx \sqrt{j/j+1}$ and the expression for the polarization of conversion electrons will assume the form

$$\langle\boldsymbol{\sigma}\rangle = \alpha\frac{j(j+1) + J_1(J_1+1) - J_2(J_2+1)}{2j(2j+1)J_1}\{(\boldsymbol{Vn})\boldsymbol{n} + j(\boldsymbol{V} - (\boldsymbol{Vn})\boldsymbol{n})\}. \tag{19}$$

### 3. – Electric transition.

Substituting in expression (3$a$) wave functions of electron $\psi_1$ and $\psi_2$ from (1) and (2) and the operator $B^{(1)}_{j\mu}$, we will get

$$(20)\quad (B^{(1)}_{j\mu})_{21} = \sum_{j_2 l_2 \mu_2} [\Omega^{(l_2)}_{j_2\mu_2}(\boldsymbol{n})]_\xi \exp[i\delta_{\varkappa_2}] \{\sqrt{j}\left[R_{3,\varkappa_2}\int \Omega^{(l_2)*}_{j_2\mu_2} u_{\mu_1} y_{j\mu}\,\mathrm{d}0 + \right.$$

$$\left. + R_{4\varkappa_2}\int \Omega^{(l'_2)*}_{j_2\mu_2}\,\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\,u_{\mu_1} y_{j\mu}\,\mathrm{d}0\right] + \sqrt{2j+1}\left[R_{,\varkappa_2}\int \Omega^{(l_2)*}_{j_2\mu_2}(\boldsymbol{\sigma}\boldsymbol{y}_{j,j-1,\mu})\left(\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\right)u_{\mu_1}\,\mathrm{d}0 - \right.$$

$$\left. - R_{5,\varkappa_2}\int \Omega^{(l'_2)*}_{j_2\mu_2}\,\boldsymbol{\sigma}\boldsymbol{y}_{j\,j-1,\mu}\,u_{\mu_1}\,\mathrm{d}0\right\}.$$

Here we introduce the following notations for radial integrals

$$(21)\quad \begin{cases} R_{3\,\varkappa_2} = \int G_j(wr) g_{\varkappa_2}(r) g(r) r^2\,\mathrm{d}r\,; & iR_{5,\varkappa_2} = \int G_{j-1}(wr) f_{\varkappa_2}(r) g(r) r^2\,\mathrm{d}r\,, \\ R_{1,\varkappa_2} = \int G_j(wr) f_{\varkappa_2}(r) f(r) r^2\,\mathrm{d}r\,; & iR_{6,\varkappa_2} = \int G_{j-1}(wr) g_{\varkappa_2}(r) f(r) r^2\,\mathrm{d}r\,. \end{cases}$$

Let us calculate the angular integrals:

$$(22)\qquad \int \Omega^{(l_2)*}_{j_2\mu_2} u_{\mu_1} y_{j\mu}\,\mathrm{d}0 = \int \Omega^{(l'_2)*}_{j_2\mu_2}\,\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\,u_{\mu_1} y_{j\mu}\,\mathrm{d}0 = C^{l_2\mu_2}_{\frac{1}{2}\mu_1;j\mu}\,\delta_{l_2 j}\,.$$

For the calculation of the second pair of angular integrals let us expand the spherical vector $\boldsymbol{y}_{j,j-1,\mu}$: into orthogonal spherical vectors $\boldsymbol{y}^{(-1)}_{j\mu}$ and $\boldsymbol{y}^{(1)}_{j\mu}$

$$(23)\qquad \boldsymbol{y}_{j,j-1,\mu} = \sqrt{\frac{j}{2j+1}}\,\boldsymbol{y}^{(-1)}_{j\mu} + \sqrt{\frac{j+1}{2j+1}}\,\boldsymbol{y}^{(1)}_{j\mu}\,,$$

and we will get

$$(24)\quad \begin{cases} \left(\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\right)(\boldsymbol{\sigma}\boldsymbol{y}_{j,j-1,\mu}) = \left(\sqrt{\frac{j}{2j+1}} - \frac{\boldsymbol{\sigma}\boldsymbol{L}}{\sqrt{j(2j+1)}}\right) y_{j\mu}\,, \\ \boldsymbol{\sigma}\left(\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\right)\boldsymbol{y}_{j,j-1,\mu} = \left(\sqrt{\frac{j}{2j+1}} + \frac{\boldsymbol{\sigma}\boldsymbol{L}}{\sqrt{j(2j+1)}}\right) y_{j\mu}\,. \end{cases}$$

For the second pair of radial integrals we will get:

$$(25)\quad \int \Omega^{(l_2)*}_{j_2\mu_2}\left(\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\right)\boldsymbol{\sigma}\boldsymbol{y}_{j,j-1,\mu}\,u_{\mu_1}\,\mathrm{d}0 = \begin{cases} 0; & j_2 = j+\frac{1}{2}, \\ \sqrt{\dfrac{2j+1}{j}}\,C^{j_2\mu_2}_{\frac{1}{2}\mu_1;j\mu}\,\delta_{l_2 j}; & j_2 = j-\frac{1}{2}; \end{cases}$$

$$(26)\quad \int \Omega^{(l_2)*}_{j_2\mu_2}\,\boldsymbol{\sigma}\left(\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}\right)\boldsymbol{y}_{j\,j-1,\mu}\,u_{\mu_1}\,\mathrm{d}0 = \begin{cases} 2\sqrt{\dfrac{j}{2j+1}}\,C^{j_2\mu_2}_{\frac{1}{2}\mu_1;j\mu}\,\delta_{l_2 j}; & j_2 = j+\frac{1}{2}, \\ -\dfrac{1}{\sqrt{j(2j+1)}}\,C^{j_2\mu_2}_{\frac{1}{2}\mu_1;j\mu}\,\delta_{l_2 j}; & j_2 = j-\frac{1}{2}. \end{cases}$$

The quantity $\varkappa_2$ may have two values

$$(27)\qquad \varkappa_2 = \begin{cases} -j-1 & j_2 = j+\frac{1}{2}, \\ j & j_2 = j-\frac{1}{2}. \end{cases}$$

Substituting the values of angular integrals in the formula (20) we will get

$$(28)\quad (B^{(1)}_{j\mu})_{21} = \sqrt{j}\left(R_{3,-j-1} + R_{4,-j-1} + 2R_{6,-j-1}\right)\exp[i\delta_{-j-1}]\,C^{j+\frac{1}{2},\mu+\mu_1}_{j\mu;\frac{1}{2}\mu_1}\cdot C^{j+\frac{1}{2},\mu+\mu_1}_{j,\mu+\mu_1-\xi;\frac{1}{2}\xi}\cdot$$

$$\cdot y_{j,\mu+\mu_1-\xi}(\boldsymbol{n}) + \sqrt{j}\left(R_{3,j} + R_{4,j} - \frac{2j+1}{j}R_{5,j} - \frac{1}{j}R_{6,j}\right)\exp[i\delta_j]\cdot$$

$$\cdot C^{j-\frac{1}{2},\mu+\mu_1}_{j\mu;\frac{1}{2}\mu_1}\,C^{j-\frac{1}{2},\mu+\mu_1}_{j,\mu+\mu_1-\xi;\frac{1}{2}\xi}\,y_{j,\mu+\mu_1-\xi}(\boldsymbol{n}).$$

Expression (28) with the accuracy up to the common factor may be rewritten in a more convenient form:

$$(29)\qquad (B^{(1)}_{j\mu})_{21} = \boldsymbol{y}^{(0)}_{j\mu}\boldsymbol{\sigma} + A^{(1)}y_{j\mu},$$

where

$$(30)\quad A^{(1)} = \frac{1}{\sqrt{j(j+1)}}\cdot$$

$$\cdot\frac{(j+1)(R_{3,-j-1}+R_{4,-j-1}+2R_{6,-j-1})+j\left(R_{3,j}+R_{4,j}-\dfrac{2j+1}{j}R_{5,j}-\dfrac{1}{j}R_{6,j}\right)\exp[j(\delta_j-\delta_{-j-1})]}{R_{3,\,j\,1}+R_{4,-j\,1}+2R_{6,-j-1}-\left(R_{3,j}+R_{4,j}-\dfrac{2j+1}{j}R_{5,j}-\dfrac{1}{j}R_{6,j}\right)\exp[i(\delta_j-\delta_{-j\,1})]}.$$

Having made the same calculations as in the case of magnetic transition, we will get for the polarization of conversion electrons:

$$(31)\quad \langle \boldsymbol{\sigma} \rangle = \alpha \frac{j(j+1)+J_1(J_1+1)-J_2(J_2+1)}{2j(j+1)J_1(1+|A^{(1)}|^2)} \{(\boldsymbol{Vn})\boldsymbol{n}+\sqrt{j(j+1)}\,\mathrm{Re}A^{(1)}(\boldsymbol{V}-(\boldsymbol{Vn})\boldsymbol{n})\}.$$

When free electrons are approaching, radial integrals are calculated in the obvious form

$$R_{3\varkappa_1} = (2\bar{n}Ze^2)^{s}\sqrt{\frac{2(\varepsilon_2+m)}{w}}\, I_j\cdot(i)^j;$$

$$R_{5,j} = -(2\bar{n}Ze^2)^{s}\sqrt{\frac{2(\varepsilon_2-m)}{w}}\, I_{j-1}\cdot(i)^j;$$

$$R_{1,\varkappa_2} = R_{6,\varkappa_2} = 0; \qquad \delta_j - \delta_{-j-1} = 0\,.$$

Substituting these values for radial integrals in (30) we will get

$$(32)\qquad A^{(1)} = -\sqrt{\frac{j}{j+1}}\,\frac{2\varepsilon_2}{\varepsilon_2-m}\,.$$

Substituting this value $A^{(1)}$ in (31) we will get the same value for polarization as in [1].

For large $Z$ radial integrals in the obvious form cannot be obtained and they should be calculated by numerical methods. Let us give for $Z=80$ a table of values $\sqrt{j(j+1)}\,A^{(1)}$ calculated with the aid of values of radial integrals, so kindly given by L. A. Sliv.

TABLE I.

| $j$ \ $\omega/m$ | 0.3 | 0.5 | 0.7 | 1 |
|---|---|---|---|---|
| 1 | 4.6 | 3.3 | 2.5 | 2.0 |
| 2 | 3.7 | 2.5 | 2.0 | 1.4 |
| 3 | 3.2 | 2.2 | 1.6 | $1.04-0.27i$ |

As can be seen from Table I, the electric field of the nucleus exerts a substantial influence upon the polarization of conversion electrons, and for electric transitions.

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If there is a mixture of electric and magnetic multipoles, then since there is a designated direction (direction of the emission of a β-electron), there is an interference of the electric and magnetic multipoles.

* * *

In conclusion we wish to express profound gratitude to Prof. V. B. Berestetsky and A. P. Rudik for the formulation of the problem and for the discussions, to Academicians A. I. Alihanov and V. A. Ljubimov for the interest shown in the work and discussions, and to Prof. L. A. Sliv, who offered the values of radial integrals.

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## RIASSUNTO (*)

Si considera la correlazione della polarizzazione degli elettroni di conversione con la direzione degli elettroni del precedente decadimento β. In tutto il calcolo si è tenuto conto del campo elettrico del nucleo. Dal calcolo del campo elettrico del nucleo risulta nelle transizioni magnetiche la comparsa di una considerevole componente trasversale della polarizzazione, mentre nelle transizioni elettriche l'intensità della polarizzazione è sostanzialmente accresciuta.

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(*) *Traduzione a cura della Redazione.*

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# Polarization of the Internal Conversion Electrons Accompanying β-Decay.

V. B. Berestetckij and A. P. Rudik

*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — The correlation of the polarization of the conversion electrons and the velocity of the electrons emitted in a previous allowed β-decay transition is considered. In the magnetic multipole case the polarization is longitudinal and energy independent if we neglect the influence of the nuclear Coulomb field. In the electric multipole case exist as well longitudinal as transversal polarization, both depending on the electron energy.

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1. – Owing to parity non-conservation in β-decay the daughter nucleus becomes polarized in the direction of the emitted β-electron. (We assume that the parent nucleus is unpolarized and that the direction of motion of the neutrino is not fixed). If the process of internal conversion takes place after β-decay, the conversion electrons must therefore possess a certain polarization (*). This effect may be used both for the investigation of β-decay and to study the properties of nuclear levels because (this will be shown below) the polarization of the conversion electrons essentially depends on the order and type (electric or magnetic) of the nuclear transition.

We shall limit ourselves to the case of allowed β-transitions. The general expression for the conversion electron polarization vector $\langle \boldsymbol{\sigma} \rangle$ must have the following form

$$\langle \boldsymbol{\sigma} \rangle = a(\boldsymbol{v}\boldsymbol{n})\boldsymbol{n} + b(\boldsymbol{v} - (\boldsymbol{v}\boldsymbol{n})\boldsymbol{n}) , \tag{1}$$

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(*) Our attention was directed to the existence of such an effect by A. Alihanov and V. Ljubimov.

where $a$ and $b$ are constants, depending on the spins and energies of the nuclear states, $\boldsymbol{v}$ is the velocity of the β-electron, $\boldsymbol{n}$ a unit vector in the direction of motion of the conversion electron. In fact, (1) is the most general expression, which has the following properties: 1) it is a polar vector; it corresponds to the fact that polarization appears only in virtue of parity non-conservation (for $\langle\boldsymbol{\sigma}\rangle$ is an axial vector); 2) it is invariant relative to transformation $\boldsymbol{n} \to -\boldsymbol{n}$, which corresponds to parity conservation in the internal conversion process; 3) it is proportional to the velocity of the β-electron, which determines the daughter nucleus' polarization.

2. – Let us find the density matrix, determining the polarization state of the nucleus after β-decay. If this nucleus has the spin value $j_2$ and the initial nucleus is unpolarized, and the neutrino direction is not fixed, the general expression for the density matrix $\varrho_{m_2 m_2'}$ determining the distribution of the spin projection $m_2$ is

$$\varrho_{m_2 m_2'} = \frac{1}{2j_2+1}\left(\delta_{m_2 m_2'} + \sqrt{\frac{j_2+1}{j_2}}\,\zeta C^{j_2 m_2}_{j_2 m_2' 1\mu} v^\mu\right), \tag{2}$$

where $v^\mu = (-1)^\mu v_\mu$ are vector components ($v^0 = v_z$; $v^{\pm 1} = \mp (1/\sqrt{2})(v_x \mp i v_y)$); $C\cdots$ the vector addition coefficients. The latter differing only by a normalizing factor from the matrix elements of the angular momentum operator $J$

$$(jm|J_\mu|jm') = (jm'|J^\mu|jm) = \sqrt{j(j+1)}\, C^{jm}_{jm'1_\mu}.$$

The constant $\zeta$ may be expressed in terms of the constant which determines the angular distribution of the β-electrons in the decay of a polarized nucleus.

Let us consider to this purpose the β-transition of a polarized nucleus with spin $j_2$ into a nucleus with spin $j_3$. The decay probability can be written as

$$W = \mathrm{Tr}\, \varrho^0 U^+ U,$$

where $U$ is the operator determining the transition $j_2 \to j_3$ and $\varrho^0$ is the density matrix of the $j_2$-state. If the polarization state is determined only by its average spin-vector $\langle \boldsymbol{J} \rangle$ we have

$$\varrho^0_{m_2 m_2'} = \frac{1}{2j_2+1}\left(\delta_{m_2 m_2'} + \frac{3\langle J_\mu\rangle}{\sqrt{j_2(j_2+1)}} C^{j_2 m_2}_{j_2 m_2' 1\mu}\right).$$

The matrix $U^+U$ is proportional to the expression (2). Actually, the density matrix (2) is determined by the $j_3 \to j_2$ transition and because of the fact

that the state $j_3$ is unpolarized

$$\varrho_{m_3 m_3} = \frac{1}{2j_3+1}\,\delta_{m_3 m_3'}\,.$$

Therefore

$$W = 1 + \zeta \frac{\langle \boldsymbol{J} \rangle}{j_2}\,\boldsymbol{U}\,. \tag{3}$$

So, we may not consider at all the β-decay stage of the process but determine the constant $\zeta$ in (2) comparing equation (3) with the known expression for $W$ (1).

For allowed β-decay neglecting Coulomb corrections we have

$$\zeta = \frac{2\,\mathrm{Re}\{C_S C_T'^{*} + C_S' C_T^{*} - C_V C_A'^{*} - C_V' C_A^{*}\}\sqrt{j_2/(j_2+1)}\,\delta_{j_2 j_3} M_F M_{GT}^{*} + (C_T C_T'^{*} - C_A C_A'^{*})\Lambda_{j_2 j_3}|M_{GT}|^2}{(|C_S|^2+|C_S'|^2+|C_V|^2+|C_V'|^2)|M_F|^2+(|C_T|^2+|C_T'|^2+|C_A|^2+|C_A'|^2)|M_{GT}|^2}\,,$$

where

$$\Lambda_{j_2 j_3} = \frac{j_2(j_2+1) - j_3(j_3+1) + 2}{(2j_2+1)j_2}\,.$$

3. – Let the nucleus pass further from the state $j_2 m_2$ into the state $j_1 m_1$. The internal conversion matrix element may be written in the following way (we omit the non-essential factors) (2)

$$m_{m_1 m_2} = (j_2 m_2 | Q_{LM}^{(\lambda)} | j_1 m_1)^{*} \int \psi_2^{*}(\boldsymbol{r}) B_{LM}^{(\lambda)}(\boldsymbol{r}) \psi_1(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\,. \tag{4}$$

Here $Q_{LM}^{(\lambda)}$ is the $2^L$-pole electric ($\lambda = 1$) or magnetic ($\lambda = 0$) moment operator of the nucleus, corresponding to the transition; $\psi_1$ and $\psi_2$ the initial and final wave functions of the electron, $B_{LM}^{(\lambda)}$ is the operator of the interaction of the electron with the multipole field. This operator has the following form

$$B_{LM}^{(0)} = \boldsymbol{\alpha} G_L(wr)\boldsymbol{Y}_{LLM}\left(\frac{\boldsymbol{r}}{r}\right),$$

$$B_{LM}^{(1)} = G_L(wr) Y_{LM}\left(\frac{\boldsymbol{r}}{r}\right) + \sqrt{\frac{2L+1}{L}}\,\boldsymbol{\alpha}\boldsymbol{Y}_{L,L-1,M}^{(r/r)}\left(\frac{\boldsymbol{r}}{r}\right) G_{L-1}(wr)\,,$$

(1) T. D. Lee and C. N. Yang: *Phys. Rev.*, **104**, 254 (1956).
(2) A. L. Ahiezer and V. B. Berestetckij: *Quantum Electrodynamics.*

where $\alpha$ is the Dirac matrix, $w$ the transition energy, $G_l(x) = i^l\left(H^{(1)}_{l+\frac{1}{2}}(x)\right)/\sqrt{x}$ ($H^{(1)}$ the Hankel-function), $Y_{LM}$ a spherical function, $Y_{LeM}$ a spherical vector with the components

$$(Y_{LlM})^{\mu} = C^{LM}_{lm1\mu} Y_{lm} .$$

We shall limit ourselves to the free electron approximation

$$\psi_2 = \begin{pmatrix} u \\ \dfrac{\boldsymbol{q\sigma}}{\varepsilon + m} u \end{pmatrix} \exp\left[i\boldsymbol{qr}\right]; \qquad \psi_1 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \quad \varepsilon = m + w ,$$

where $\varepsilon$ and $\boldsymbol{q}$ are the energy and momentum of the conversion electron, $u$ and $u_0$ two component spinors, $\boldsymbol{\sigma}$ the Pauli matrix. Then the integral containing in (4) reduces to the following

$$\int \exp\left[-\boldsymbol{qr}\right] Y_{lm}\left(\frac{\boldsymbol{r}}{r}\right) G_l(wr)\, \mathrm{d}\boldsymbol{r} \sim Y_{lm}(\boldsymbol{n})\left(\frac{q}{w}\right)^l \qquad (\boldsymbol{n} = \boldsymbol{q}/q) .$$

Omitting non-essential factors we obtain the expression for the matrix element

$$m_{m_2 m_1} = (j_2 m_2 / Q^{(\lambda)}_{LM} / j_1 m_1)\, U^* V^{(\lambda)}_{LM} U_0 \qquad (\boldsymbol{n} = \boldsymbol{q}/q), \tag{5}$$

where in the case of the magnetic multipole ($\lambda = 0$)

$$V^{(0)}_{LM} = (\boldsymbol{\sigma n})\boldsymbol{\sigma}\, \boldsymbol{Y}_{LLM}(\boldsymbol{n}) \tag{5a}$$

and in the case of the electric multipole ($\lambda = 1$)

$$V^{(1)}_{LM} = Y_{LM}(\boldsymbol{n}) + (\boldsymbol{\sigma n})\boldsymbol{\sigma}\, \boldsymbol{Y}^{(\boldsymbol{n})}_{L,L-1,M} \sqrt{\frac{2L+1}{L}}\, \varkappa \qquad \left(\varkappa = \frac{\varepsilon - m}{\varepsilon + m}\right), \tag{5b}$$

The internal conversion probability is determined by the quantity

$$W = \varrho_{m_2 m_2'} m_{m_2 m_1} m^*_{m_2' m_1} . \tag{6}$$

If it is represented in the form

$$W = \mathcal{P}_{\alpha\beta} U_\alpha U^*_\beta ,$$

then the density matrix of the conversion electron will be

$$\mathcal{P}/\mathrm{Tr}\,\mathcal{P}\,.$$

If further $\mathcal{P}$ has the form

$$\mathcal{P} = A(1+\boldsymbol{\xi}\boldsymbol{\sigma}) \tag{6'}$$

the conversion electron polarization vector is equal

$$\langle\boldsymbol{\sigma}\rangle = \boldsymbol{\xi}\,.$$

4. – Substituting (5) into (6) we will make use of the fact that the initial electron state is unpolarized and therefore $u_\alpha \cdot u^*_\beta \to \delta_{\alpha\beta}$. Then we shall obtain the following expression for $\mathcal{P}$

$$\mathcal{P} = \varrho_{m_2 m_2'}(j_2 m_2|Q^{(\lambda)}_{LM}|j_1 m_1)(j_2 m_2'|Q^{(\lambda)}_{LM'}|j_1 m_1)V^{(\lambda)}_{LM}V^{(\lambda)*}_{LM'}\,. \tag{7}$$

The product of the last two terms being a density matrix relative to spin variables may be represented as

$$V^{(\lambda)}_{LM}V^{(\lambda)}_{LM'} = R^{(\lambda)}_{MM'} + \boldsymbol{\sigma}\boldsymbol{S}^{(\lambda)}_{MM'}\,. \tag{8}$$

Using the expressions (5*a*) and (5*b*) we obtain

$$\left\{\begin{aligned} &R^{(0)}_{MM'} = \boldsymbol{Y}_{LLM}Y^*_{LLM'};\ \ \boldsymbol{S}^{(0)}_{MM'} = i\,[\boldsymbol{Y}_{LLM}\boldsymbol{Y}^*_{LLM'}]\,,\\ &R^{(1)}_{MM'} = (1+2\varkappa)Y_{LM}Y^*_{LM'} + \varkappa^2\,\frac{2L+1}{L}\,\boldsymbol{Y}_{LL-1M}\boldsymbol{Y}^*_{LL-1M'}\,,\\ &\boldsymbol{S}^{(1)} = i\varkappa^2\,\frac{L+1}{L}\,[\boldsymbol{Y}_{LLM}\boldsymbol{Y}_{LLM'}] - \varkappa(1+\varkappa)\sqrt{\frac{L+1}{L}}\,(\boldsymbol{Y}_{LLM}Y^*_{LM'} + Y_{LM}\boldsymbol{Y}^*_{LLM'})\,. \end{aligned}\right. \tag{8}$$

The multipole moment matrix element can be represented as following

$$(j_2 m_2|Q^{(\lambda)}_{LM}|j_1 m_1) = Q^{(\lambda)}\,C^{j_2 m_2}_{j_1 m_1 LM}\,, \tag{9}$$

where $Q^{(\lambda)}$ does not depend on the quantum numbers $m_2$ and $m_1$.

Substituting (2), (8) and (9) into (7) we obtain, omitting non-essential factors

$$\mathcal{P} = \left(\delta_{m_2 m_2'} + \sqrt{\frac{j_2+1}{j_2}}\,\zeta C^{j_2 m_2}_{j_2 m_2 1M}\,v^\mu\right)C^{j_2 m_2}_{j_1 m_1 LM}\,C^{j_2 m_2'}_{j_1 m_1 LM}(R^{(\lambda)}_{MM'} + \boldsymbol{\sigma}\boldsymbol{S}^{(\lambda)}_{MM'})\,,$$

a summation in all related indices is implied, and using the equations

$$C^{j_2 m_2}_{j_1 m_1 LM} C^{j_2 m_2}_{j_1 m_1 LM'} = \delta_{MM'}(2j_2+1)/(2L+1)\,,$$

$$C^{j_2 m_2}_{j_1 m_1 LM} C^{j_2 m_2}_{j_1 m_1 LM'} C^{j_2 m_2}_{j_2 m_2 1\mu} = (2j_2+1)/(2L+1)\cdot \frac{L(L+1)+j_2(j_2+1)-j_1(j_1+1)}{2\sqrt{L(L+1)j_2(j_2+1)}}\, C^{LM}_{LM1\mu}\,,$$

we have

$$\mathcal{P} = (R^{(\lambda)}_{MM'} + \boldsymbol{\sigma} \boldsymbol{S}^{(\lambda)}_{MM'})\cdot \left(\delta_{MM'} + \sqrt{\frac{j_2+1}{j_2}}\,\zeta\, \frac{L(L+1)+j_2(j_2+1)-j_1(j_1+1)}{2\sqrt{L(L+1)j_2(j_2+1)}}\, C^{LM}_{LM'1\mu} v^{\mu}\right). \tag{10}$$

The first two terms in (10) cannot depend on $n$ after the summation over $M$ is carried out. Therefore $\boldsymbol{S}^{(\lambda)}_{MM} = 0$ and $R^{(\lambda)}_{MM} = (1/4\pi)\int R^{(\lambda)}_{MM}\,\mathrm{d}0$. It follows from (8)

$$R^{(0)}_{MM} = (2L+1)/4\pi\,,$$

$$R^{(1)}_{MM} = \left(1 + 2\varkappa + \frac{2L+1}{L}\varkappa^2\right)(2L+1)/4\pi\,.$$

To compute the last two terms in (10) let us choose the $Z$-axis along the vector $\boldsymbol{v}$. Then, instead of a sum over $\mu$ one term with $\mu = 0$ remains, and $M' = M$. Using the expression for the coefficient

$$C^{LM}_{LM10} = \frac{M}{\sqrt{L(L+1)}},$$

and taking into account that $R^{(\lambda)}_{MM}$ doesn't depend on the sign of $M$ we obtain

$$\sum_M M R^{(\lambda)}_{MM} = 0\,.$$

Let us consider now the last term in (10), *i.e.* the quantity

$$v C_{LM10} \boldsymbol{S}^{(\lambda)}_{MM}\,. \tag{11}$$

In the magnetic multipole case the vector $\boldsymbol{S}^{(0)}$ according to (8) has the direction of $\boldsymbol{n}$ for $\boldsymbol{Y}_{LLM}$ is a transversal vector. That means that the coefficient $b$ in (1) is absent in magnetic transitions.

Comparing (11) with the first term in (1) we see that

$$v M \boldsymbol{S}^{(0)}_{MM} = f(\boldsymbol{vn})\boldsymbol{n}\,, \tag{12}$$

where $f$ is a constant.

For its determination let us integrate (12) over the angles, which gives

$$\frac{L\pi}{3} f = \sum_M M \int S^{(0)}_{ZMM} \, \mathrm{d}0 \, .$$

Substituting the expression for $S^{(0)}_{MM}$ we find $f = (2L+1)/4\pi$. In the electric multipole case $S^{(1)}$ according to (8′) consists of two terms. The first term has the direction $\boldsymbol{n}$ and can be computed in the same way as $S^{(0)}$. The second term, which is proportional to $Y_{LLM}$ is perpendicular to $\boldsymbol{n}$ and, consequently must have the form of the second term in (1), *i.e.*

$$g(\boldsymbol{v} - (\boldsymbol{vn})\boldsymbol{n}) \, .$$

Using the expression for $S^{(1)}$ and integrating this equation over the angles we find

$$g = \frac{2L+1}{4\pi} \sqrt{L(L+1)} \, .$$

Substituting these results into (10) we obtain with (6′) and (6″) the following expressions for the polarization vector of the conversion electron:

in the magnetic multipole case

$$\langle \boldsymbol{\sigma} \rangle = \frac{\eta}{j_2} \zeta \boldsymbol{n}(\boldsymbol{nv}) \, , \tag{13}$$

in the electric multipole case

$$\langle \boldsymbol{\sigma} \rangle = \frac{(L+1) r_\zeta^\varkappa}{j_2(1 + 2\varkappa + \varkappa^2(2L+1)/L)} \left\{ (\varkappa + \varkappa^2)(\boldsymbol{n}(\boldsymbol{nv}) - \boldsymbol{v}) + \varkappa^2 \boldsymbol{n}(\boldsymbol{nv}) \right\} , \tag{14}$$

where

$$\eta = \frac{L(L+1) + j_2(j_2+1) - j_1(j_1+1)}{2L+1} \, .$$

We see from (13) that the polarization in the magnetic multipole case is longitudinal and does not depend on the conversion electron energy. But this property is closely connected to the free electron approximation used here. Therefore a consideration of this problem with the exact electron wave function is of interest.

In the electric multipole case the polarization according to (14) depends in a considerable degree on the energy. If the conversion electron velocity is

small the transversal polarization is of order $v^2/c^2$ and the longitudinal polarization — $v^4/c^4$. These results are not precise because of the significance which the Coulomb field may have at small velocities.

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We express our sincere gratitude to acad. A. ALIHANOV and V. LJUBIMOV for their interest in our work and useful discussions.

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## RIASSUNTO (*)

Si considera la correlazione della polarizzazione degli elettroni di conversione con la velocità degli elettroni emessi un precedente decadimento β permesso. Nel caso di multipolo magnetico la polarizzazione è longitudimale e indipendente dall'energia se si trascura l'influenza del campo coulombiano nucleare. Nel caso di multiplo elettrico si ha polarizzazione sia longitudinale che traversale ambedue dipendenti dall'energia degli elettroni.

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# On β-Transitions 0—0 with Change of Parity.

B. V. Geškenbein

*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — It is shown that the form of the spectrum and the polarization of the electrons in 0—0 β transitions is in agreement with a $V$-$A$ type of interaction.

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Transition 0—0 with change of parity is one of the simple forbidden β-transitions. In these transitions contribution may be made only by tensor, pseudo-scalar and axial-vector variants of interaction. Scalar and vector variants of interaction are forbidden by the selection rules. As is known [1], the spectrum 0—0 (yes) transitions with a great degree of accuracy coincide with the Fermi, *i.e.* correction factor $C$ describing the deviation of the form of the spectrum from the form of the allowed spectrum, is independent of energy. The tensor variant of interaction alone does not give the necessary form of the spectrum and for an explanation of the form of the spectrum it is necessary to involve as well the pseudo-scalar variant of the interaction, and it was required that $g_P \gg g_T$. Recently a universal scheme [2,3] for all the weak interactions was proposed. The β-decay according to this scheme may be only $A$ and $V$ variants of interaction.

The aim of the present report is to call attention to the fact that the form of the spectrum 0—0 (yes) transition is fully in accord with the $A$-variant of interaction, and with the present formulae for the polarization of the decay electrons and of the angular electron-neutrino correlation.

The desired formulae are derived from the formulae for $T$-$P$ variants of

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(1) L. N. Ziryannova: *Izv. Acad. Sci. USSR, Phys. Ser.*, **20**, 1399 (1956).
(2) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).
(3) E. C. G. Sudarshan and R. E. Marshak: Preprint.

interaction [4] if we substitute in them $\lambda_P = ig_P \int \gamma_5 / g_T \int \boldsymbol{\sigma r}$ into $\lambda = -i \int \gamma_5 / \int \boldsymbol{\sigma r}$ which may be then considered as real and if we substitute $q$ for $-q$ ($q$ is the neutrino momentum).

The correction factor $\boldsymbol{C}$ has the form:

$$(1) \qquad \boldsymbol{C} = \left\{ (\tfrac{1}{9} L_0 q^2 + \mu_0 - \tfrac{2}{3} q N_0) + 2(N_0 - \tfrac{1}{3} L_0 q)\lambda + L_0 \lambda^2 \right\} |g_A \int \boldsymbol{\sigma r}|^2 .$$

For $^{144}$Pr the expression in the first brackets, for a change of energy from 100 keV to 2800 keV, changes only 5% (a similar expression for $T$-$P$ variant was doubled), the expression in the second parenthesis changes 3% (in the case of the $T$-$P$ variant a similar expression changed 50%). If it be required that the correction factor $\boldsymbol{C}$ with a 5% accuracy be constant, then one of the following conditions must be fulfilled by $\lambda$:

$$(2) \qquad \lambda < 3 \quad \text{or} \quad \lambda > 24 .$$

The sign of $\lambda$ is unknown.

For polarization of electron decay and for electron-neutrino correlation we have the following formulae:

$$(3) \qquad \langle \sigma n \rangle = -\frac{1}{C} |g_A \int \boldsymbol{\sigma r}|^2 \left\{ \frac{1}{9} q^2 \sqrt{L_0^2 - P_0^2} + \sqrt{M_0^2 - Q_0^2} + \right.$$

$$+ \frac{1}{3} q \left( \sqrt{(L_0 + P_0)(M_0 + Q_0)} + \sqrt{(L_0 - P_0)(M_0 - Q_0)} \right) - \left( \sqrt{(L_0 + P_0)(M_0 + Q_0)} + \right.$$

$$\left. \left. + \sqrt{(L_0 - P_0)(M_0 - Q_0)} + \frac{2}{3} q \sqrt{L_0^2 - P_0^2} \right) \lambda + \sqrt{L_0^2 - P_0^2}\, \lambda^2 \right\} \sin(\delta_{-1} - \delta_1) ,$$

$$(4) \qquad W_{e\nu}(\theta) = 1 + \langle \boldsymbol{\sigma n} \rangle \cos\theta .$$

If $\lambda$ satisfies one of the inequalities (2), then the value of polarization will practically not differ from $v/c$.

* * *

In conclusion we wish to express our most sincere thanks to Prof. V. B. Berestetsky, B. L. Ioffe and A. P. Rudik for their most valuable discussions.

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[4] B. V. Geškenbein: *Žurn. Èksper. Teor. Fiz.*, **33**, 1353 (1957).

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## RIASSUNTO (*)

Si dimostra che la forma dello spettro e la polarizzazione degli elettroni nelle transizioni 0—0 β si accorda con un'interazione di tipo $V$-$A$.

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(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 2 16 Ottobre 1958

# Nuclear Polarization in Radiative K-Capture.

V. B. Beresteckij

*Academy of Sciences - Moscow*

(ricevuto il 24 Luglio 1958)

**Summary.** — A formula for the nuclear polarization in the process $e+p \to n+\nu+\gamma$ is given.

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It is well known that owing to parity non-conservation a nucleus becomes polarized after β-decay. The average spin vector of the daughter nucleus is

$$\langle \boldsymbol{J} \rangle = \tfrac{1}{3}(j+1)\zeta \boldsymbol{v} \,, \tag{1}$$

where $j$ is its spin value, $c\boldsymbol{v}$ the velocity of the electron (the neutrino motion direction is not fixed), $\zeta$ the coefficient in the formula

$$W = 1 + \zeta \boldsymbol{v} \langle \boldsymbol{J} \rangle / j$$

determining the angular distribution of β-electrons in the decay of polarized nuclei [1,2].

This polarization leads to the known effects in the electromagnetic transitions of the daughter nucleus: circular polarization of photons [3] or polarization of conversion electrons [4].

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[1] T. D. Lee and C. N. Yang: *Phys. Rev.*, **104**, 254 (1956).

[2] V. B. Beresteckij, B. L. Ioffe, A. P. Rudic and K. A. Ter-Martirosjan: *Nuclear Physics*, **5**, 464 (1958); *Phys. Rev.* (in press); I. M. Šmuškevič: *Zu. Éksper. Teor. Fiz.*, **33**, 1476 (1957).

[3] K. Alder, B. Stech and A. Winter: *Phys. Rev.*, **107**, 728 (1957).

[4] V. B. Beresteckij and A. P. Rudik: *Zu. Éksper. Teor. Fiz.* (in press).

A similar nuclear polarization takes place also in the radiative capture of an orbital electron, *i.e.* in the process $e+p \to n+\nu+\gamma$. The nuclear polarization is determined in this case by the same formula (1), where $cv$ is now the photon velocity and $\zeta$ the coefficient related to the positron decay case.

To prove this let us consider the matrix element of the radiative K-electron capture in which the nucleus passes from a state $j_1$, $m_1$, to a state $j$, $m$. Omitting a factor

$$U_{m_1 m} = (jm\,|\,O_i\,|\,j_1 m_1)(\bar{u}_\nu(q) O_i(\hat{p} - \hat{k} + im)\hat{e}\, u_e(p)) ,$$

where $(jm\,|\,O_i\,|\,j_1 m_1)$ is the nuclear matrix element, $u_\nu(q)$ the neutrino amplitude, $q$ the neutrino momentum, $e$ the photon polarization vector, $k$ the photon momentum. The polarization density matrix of the daughter nucleus is (omitting a normalizing factor)

$$\varrho_{mm'} = \sum_{m_1} (jm\,|\,O_i\,|\,j_1 m_1)(jm' O_k\,|\,j_1 m_1)^* \operatorname{Tr} Q , \tag{2}$$

where

$$\operatorname{Tr} Q = \operatorname{Tr} (\hat{p} + im)\hat{e}\,(\hat{p} - \hat{k} + im)\,\bar{O}_k\,\hat{q}\,O_i(\hat{p} - \hat{k} + im)\hat{e} \qquad (\bar{O} = \gamma_4 O^+ O).$$

Since for a $K$-electron $\hat{p} = \gamma_4 p_4$ we obtain

$$\operatorname{Tr} Q = \operatorname{Tr} \hat{k}\,\bar{O}_k\,\hat{q}\,O_i . \tag{3}$$

In the case of the positron decay the density matrix $\varrho_{mm'}$ has the same form (2) with

$$\operatorname{Tr} Q = \operatorname{Tr} (\hat{p}_+ + im)\bar{O}_n \hat{q}\, O_i , \tag{4}$$

where $p_+$ is the momentum of the positron. In the extreme relativistic case ($p_+^2 = k^2 = 0$) the equation (4) coincides with (3); that proves the applicability of the formula (1) for the radiative capture case. Here we have the same relation between the photon associated with $K$-capture and the β-positron as in formulas determining the photon circular polarization [5].

It seems strange at first sight that the pseudovector $\langle \boldsymbol{J} \rangle$ is proportional to the photon momentum vector, while photon emission conserves parity. But actually $\langle \boldsymbol{J} \rangle$ is determined by the polarization of the captured virtual

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[5] R. E. Cutosky: *Phys. Rev.*, **107**, 330 (1957).

electron. This electron is polarized along its momentum and the last is just opposite to the photon momentum.

* * *

I wish to thank A. ALIHANOV, V. LJUBIMOV and L. OKUN for discussions.

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RIASSUNTO (*)

Si dà una formula per la polarizzazione nucleare nel processo $e+p \to n+\nu+\gamma$.

(*) *Traduzione a cura della Redazione.*

# NOTE TECNICHE

## Photometric Analysis of the Tracks in the Nuclear Emulsions.

### III. – Effect of the Development on the Photometric Width.

P. G. Bizzeti, M. G. Dagliana, M. Della Corte and L. Tocci

*Istituto Nazionale di Fisica Nucleare - Sottosezione di Firenze*
*Istituto di Fisica dell'Università - Arcetri (Firenze)*

(ricevuto il 17 Luglio 1958)

**Summary.** — The dependence of the photometric width of a track on the degree of development has been investigated in G-5 nuclear plates, developed by the standard procedure. It is found that the width varies with the development only through an additive term, which is independent of the nature and velocity of the particle which produced the track.

In a previous note I (1) a method for the photometric analysis of the tracks in nuclear emulsions has been described. In this method the photometric profile across the track is measured; its width at half height is defined as the « photometric width » $\lambda$ and, being a function of the charge $Ze$ and the speed $\beta c$ of the particle, it is taken as a suitable characteristic parameter to be used in the identification of the particle.

The parameter $\lambda$, however, is not a function of $\beta$ and $Z$ only but it also depends, to a certain extent, on the dip of the track and on the degree of development (let alone, of course, the emulsion type).

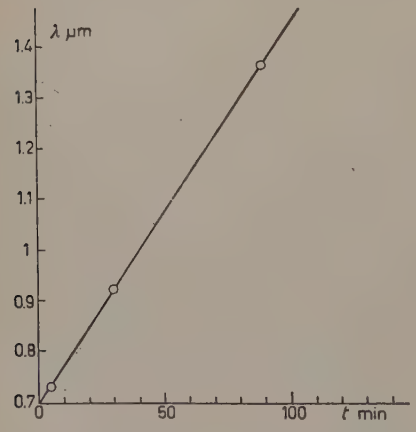


Fig. 1. – Photometric width of tracks of Po $\alpha$-particles on Ilford G-5 emulsions developed with usual ID 19 developer as a function of the development time.

(1) M. Della Corte: *Nuovo Cimento*, **4**, 1565 (1956).

The correction required for dipping tracks has been investigated in note II ([2]).

The importance of the development on the photometric width is clearly shown by Fig. 1.

In the present paper the dependence of $\lambda$ on the degree of development is investigated; this is particularly important in order to be able to compare results of measurements carried out on plates with different degrees of development.

## 1. – Considerations on the development effect suggested by current theory.

Let us consider the latent image of a track, that is the ensemble of the grains made developable by the particle. In a given type of emulsion all the characteristics of the latent image will obviously depend only on the nature and the energy of the particle. As it is currently admitted ([3-5]), the effect of development will consist not only in a blackening of the crystals, but also in a growing of them. As $\delta$-rays are not energetic enough to show up as individual tracks, and since the developable crystals are supposed to lie very close together, the track will show a quite regular contour and it will be reasonable to assume that the effect of the development on $\lambda$ consists in increasing it by an amount which does not depend on the characteristics $(\beta, Z)$ of the particle which produced the latent image.

In other words, in these conditions, it should be possible to split the observed width in two parts, the first one — $\lambda_0$ — dependent on the degree of development only, the second one — $\lambda_1(\beta, Z)$ — dependent on $Z$ and $\beta$ only ([6]), so that

$$\lambda = \lambda_0 + \lambda_1(\beta, Z) . \tag{1}$$

If a split of this kind is actually possible, we shall expect that the plot of $\lambda$ against $\beta$, corresponding to the same particle, but in plates with different development, shall overlap by simple translation along the $\lambda$ axis.

Besides, the difference between the values of $\lambda$ pertaining to two particles with given but different $Z$ should be a function of $\beta$ but not of the degree of development.

## 2. – Experimental results.

For the experimental check of the above assumptions we used three G-5 plates henceforth denoted as $L$, $F$ and $S$.

Both $L$ and $F$ are 200 $\mu$m thick and were exposed to a $^{12}C$ ion beam (about 100 MeV, with an admixture of some $\alpha$'s of the same $H\varrho$), but they markedly differ in the degree of development. The dip (in the shrinked emulsion)

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([2]) P. G. Bizzeti and M. Della Corte: *Nuovo Cimento*, **7**, 231 (1958).

([3]) M. Della Corte, M. Ramat and L. Ronchi: *Nuovo Cimento*, **10**, 509 (1953).

([4]) C. O'Ceallaigh: *C. R. Congr. Inter. Ray Cosmiques* (Bagnère de Bigorre, 1953).

([5]) A. J. Hertz and G. Davis: *Austr. Journ. Phys.*, **8**, 139 (1955).

([6]) P. G. Bizzeti and M. Della Corte: to appear shortly in *Il Nuovo Cimento*.

of the tracks was the same for both $\alpha$ and C ions, and had a value of 6% for plate $L$ and of 18% for plate $F$.

Plate $S$ was a 600 $\mu$m thick G-5 exposed to high altitude cosmic rays in the Sardinia expedition; on this plate only protons and $\alpha$'s and a few heavier tracks, which had been selected for an another work in progress in this laboratory, have been analyzed.

Some of them were identifiable by other methods.

In Fig. 2 the values of $\lambda$ as a function of $\beta$ are plotted for $\alpha$-particles in the three plates: measurements were carried out with a slit of 12.7 $\mu$m equivalent width; successive non overlapping cells were used.

It is apparent that the three curves within the limits of experimental error, can be made to overlap by a simple translation. The last three points for plate $F$ show an anomalous behaviour, which can be traced back to a strong gradient of development near the surface.

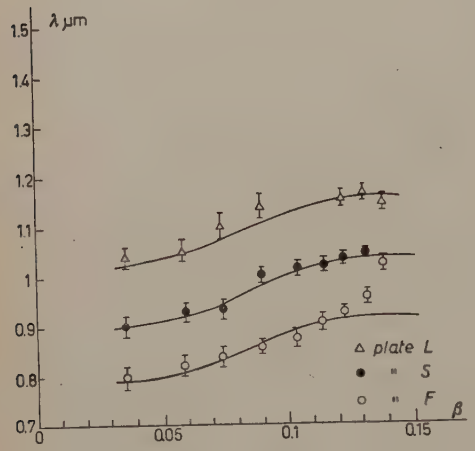


Fig. 2. – Photometric width of tracks of $\alpha$-particles as a function of $\beta$. The curves correspond to Ilford G-5 plates with different degree of development.

A more stringent test could be made by comparing the width of $^{12}C$ ions in the two plates $L$ and $F$, because the relative importance of $\lambda_1(\beta, Z)$ is higher for these ions and any dependence of $\lambda_1$ on the degree of development would show up more clearly.

It was not possible, unfortunately, to make a direct comparison of this type, owing to a quite appreciable gradient of development in plate $F$, near the surface where the Carbon tracks actually lie.

It was therefore decided to check our assumption (1), by taking the difference of the width of carbon ion and $\alpha$-tracks, corresponding to the same depth in the emulsion and therefore to the same degree of development.

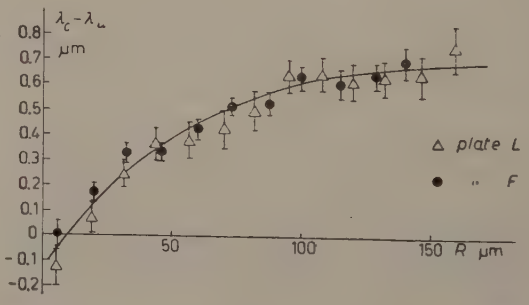


Fig. 3. – Difference $\lambda_c - \lambda_\alpha$ for the two plates $F$ and $L$, for different values of the residual range of Carbon ions.

It is worthwhile to point out that a given value of $\beta_c$ corresponds to two different depths in the two plates, but the values of $\beta_\alpha$ corresponding to these different depths are the same, so that the same value of $\beta_\alpha$ is associated (in both plates) to a given value of $\beta$. In any case the values of $\beta_\alpha$ which we are here concerned with, fall in a comparatively small range (0.12 to 0.15) and no serious error would result in considering it constant.

By plotting this difference as a function of the residual range of carbon ions we should expect on the basis of (1) that the experimental values of $\lambda_c - \lambda_x$ for plate $L$ and plate $F$ should fall on the same curve, in spite of the development being different for plate $L$ and plate $F$ and at different depth in plate $F$. Fig. 3 shows that this is actually the case, thus showing that the effect of development cancels out by taking differences and therefore giving support to our views as summarized in eq. (1), with $\lambda_0$ dependent on the development only.

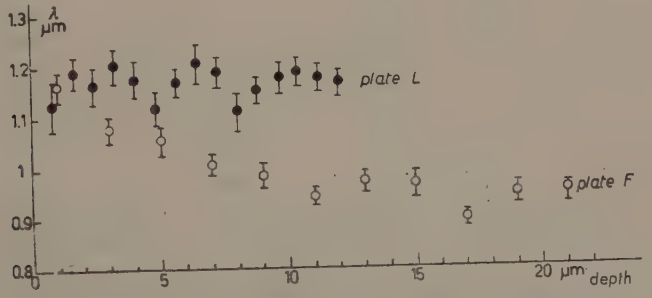


Fig. 4. – Photometric width of $\alpha$-tracks as a function of depth in the emulsions for the two plates $L$ and $F$. The development gradient in plate $F$ is clearly shown.

Fig. 4 finally shows that the contribution of the development was fairly strong, so that its cancellation in Fig. 3 is really significant. It is apparent in Fig. 4 that, while for plate $L$ the measured width of $\alpha$-tracks is nearly the same throughout the explored depth, for plate $F$ there are differences up to 25 %.

## 3. – Conclusion.

To sum up, the experimental evidence supports the view that the contribution of the development to the width of the tracks can be fairly large, but can be accounted for by a constant term (not dependent on $\beta$ and $Z$). Therefore this contribution can be easily eliminated, provided, only, some identified tracks (no matter of which type) are available in the same conditions of development.

Finally the present results can lead to a deeper insight into the physical meaning of the parameter $\lambda$ and to a better understanding of some related phenomena, as, for instance the « thinning down » effect. Work is in progress in this laboratory on the latter subject.

* * *

We are indebted to Prof. J. H. FREMLIN of the University of Birmingham for having supplied plate *F*.

Plate *L* was sent to us by Prof. J. P. LONCHAMP of the University of Strasbourg to whom we are much obliged.

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## RIASSUNTO

È stata presa in esame la dipendenza della larghezza fotometrica delle tracce dello sviluppo, in emulsioni Ilford G-5 sviluppate col procedimento standard. Sono state esaminate tracce di ioni Carbonio e di particelle $\alpha$ reperite in lastre sviluppate differentemente. I risultati sperimentali sono in accordo con una espressione del tipo $\lambda = \lambda_0 + \lambda_1(\beta, Z)$, dove la dipendenza dello sviluppo è localizzata nel termine $\lambda_0$ indipendente da $\beta$ e da $Z$.

# LETTERE ALLA REDAZIONE

*(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)*

## Rapid Decrease of Cosmic Ray Intensity.

J. F. Steljes and H. Carmichael

*Chalk River Laboratories - Ontario*

(ricevuto il 18 Luglio 1958)

Palmeira and Williams at Cambridge, Mass., U.S.A. [1] using a very high counting rate meson monitor (640 counts per second) near sea level, have reported a small but statistically significant cosmic ray decrease, beginning at 0100 UT on 11 February, 1958, some 20 to 25 min before the sudden commencement of a magnetic storm, and lasting for about $1\frac{3}{4}$ hours when a sharp Forbush type decrease ensued (*) at 0245.

The readings obtained from each of two neutron monitors in the Deep River Station (B211), also near sea level, appear to confirm this new effect although the counting rates are very much smaller. But an advantage enjoyed by the neutron monitor is that fractionally the effect as measured by neutrons is twice as large.

One of the Deep River monitors is a standard lead and paraffin neutron monitor containing 16 counters each of active volume 27 in. long by $2\frac{1}{2}$ in. in diameter filled with $^{10}BF_3$ at a pressure of 56 cm Hg at 0 °C. The other is a spherical ion chamber, 30 in. in diameter, filled with $^{10}BF_3$ at a pressure of 10 cm Hg at 0 °C, and enclosed within a graphite cube of side 8 feet. Both instruments have 5 min readout of the number of counts.

Table I shows the size of the effect in each of the three monitors. Three intervals of one hour each have been selected; 0000 to 0100, just before the effect began in any of the monitors; 0120 to 0220 following the time of the small decrease; and 0400 to 0500 after the main decrease was over. The totals counted in the first hour, the percentage decreases and the overall standard deviations of the decreases are given. The fact that two independent neutron monitors behaved similarly strengthens the probability that the effects seen are real.

In Fig. 1, the combined count of the two neutron monitors over 10 minute intervals is shown in comparison with the M.I.T. meson count over 8 minute intervals. An obvious error in their table at 0058 has been corrected. The two graphs are plotted using the natural logarithms of the number of counts as has been recommended by M. Wada [2]

(*) The possibility exists, of course, that the two decreases are not connected but that the earlier one is a small Forbush decrease produced by a separate emission from the sun and its coincidence with the sudden commencement of the magnetic storm may be fortuitous.

(1) R. A. R. Palmeira and R. W. Williams: *Nuovo Cimento*, **8**, 352 (1958).

(2) M. Wada: *Journ. Sci. Res. Inst.* (Japan), **51**, 201 (1957).

TABLE I. – *Comparison of three monitors for selected intervals of one hour duration.*

| February 11, 1958 | Number of counts | Percentage decrease | |
|---|---|---|---|
| Time U.T. | 0000÷0100 | 0120÷0220 | 0400÷0500 |
| Meson Scintillator (M.I.T.) | $2.3068 \cdot 10^6$ | $0.66 \pm 0.1$ | $2.27 \pm 0.1$ |
| Neutron Counters | $5.283 \cdot 10^4$ | $1.03 \pm 0.6$ | $5.46 \pm 0.6$ |
| Neutron Ion Chamber | $5.902 \cdot 10^4$ | $1.10 \pm 0.6$ | $5.98 \pm 0.6$ |

but they are arbitrarily placed in relation to each other in the vertical direction. However at any point on the ordinate before the sudden commencement of the magnetic storm instead of 23 min indicated by the meson record.

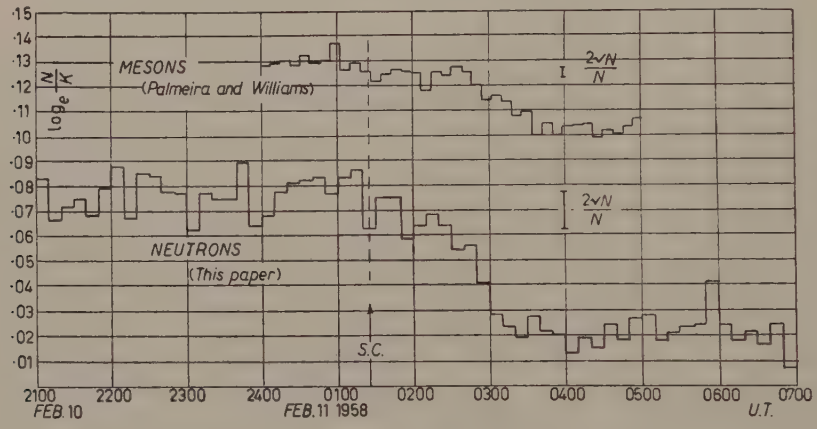


Fig. 1. – $N$ is the number of mesons counted in 8 min or the number of neutrons counted in 10 min; $\ln K$ is a constant of convenient value subtracted from $\ln N$ and is not the same for the two separate curves. Since, when $x$ is small, $\ln(1+x) = x$, a small increment of the ordinate scale equals the corresponding fractional change of the local rate. The ordinate scale divisions (0.01) are therefore approximately equal to a change of one percent in the local rate.

scale a change of one division (0.01) along the ordinate scale is approximately equal to one percent of the local rate.

The time of the sudden commencement is marked at 0125, the time given by our own magnetometer and by most of the regular observing stations ([3]). The statistical accuracy of the readings is also displayed.

The curves seem to indicate that

*a*) The preliminary neutron decrease (seen sharply also in the 5 min totals) appears to begin only 5 min

*b*) The main neutron decrease appears to be complete about 30 min sooner than the main meson decrease.

*c*) The overall neutron decrease, fractionally, is twice as large as the meson decrease.

Other stations, particularly those using neutron monitors at high altitude, may have been sufficiently powerful to give useful detail of this event. Since the whole neutron effect occurs in less than two hours little can be learned from the IGY standard bihourly totals.

([3]) *Journ. Geophys. Res.*, **63**, 418 (1958).

IL NUOVO CIMENTO Vol. X, N. 2 16 Ottobre 1958

# Note on Time-Dependent Perturbation Theory.

I-Tung Wang

*Department of Physics, University of Toronto - Toronto*

Ta-You Wu

*Division of Pure Physics, National Research Council - Ottawa*

(ricevuto il 5 Agosto 1958)

We would like to point out that the formulation of Lippmann and Schwinger [1] for the adiabatic perturbation can be generalized for a general class of time-dependent Hamiltonians. We shall first deal with the case where the perturbation $H_1(\boldsymbol{r}, t)$, continuous or piecewise continuous in time $t$, tends to zero as $t \to \pm \infty$. The wave equation in the interaction representation can be written as ($\hbar = 1$)

$$i(\partial \Psi(t)/\partial t) = H_1(\boldsymbol{r}, t)\Psi(t) ,$$

where,

$$H_1(\boldsymbol{r}, t) = \exp [iH_0 t] H_1(\boldsymbol{r}, t) \exp [-iH_0 t] , \tag{1}$$

$H_0$ being the unperturbed Hamiltonian.

The unitary operators $U(t)$ and $R$ defined by

$$\Psi(t) = U(t)\Psi(-\infty) , \qquad \Psi(\infty) = R\Psi(-\infty) ,$$

satisfy the integral equations,

$$U(t) = 1 - i \int_{-\infty}^{t} H_1(t') U(t') \, \mathrm{d}t' , \tag{2}$$

$$R = 1 - i \int_{-\infty}^{\infty} H_1(t) U(t) \, \mathrm{d}t . \tag{3}$$

(1) B. A. Lippmann and J. Schwinger: *Phys. Rev.*, **79**, 469 (1950).

The probability amplitude of a transition from $\Phi_a$ to $\Phi_b$, both eigenfunctions of $H_0$, but not necessarily belonging to the same energy shell, is

$$T_{ba} = (\Phi_b, (R-1)\Phi_a) = -i(\Phi_b, \Psi(E_a, E_b)) ,$$

where $\Psi(E_a, E_b)$ can be expressed through Eqs. (2) and (3) as

$$4) \qquad \Psi(E_a, E_b) = \int_{-\infty}^{\infty} \mathrm{d}t H_1(\boldsymbol{r},t) \exp[i(E_b - E_a)t]\, \Phi_a - \\ - i \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' H_1(\boldsymbol{r}, t) \exp[i(E_b - H_0)t] H_1(t') U(t') \Phi_a .$$

The second term on the right-hand side of Eq. (4) can be transformed into an integral involving $\Psi$, by making the Fourier transform of $H_1$,

$$(5) \qquad H_1(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} F(\boldsymbol{r}, p) \exp[ipt]\, \mathrm{d}p .$$

The integration carried out separately in three quadrants of the $t - t'$ plane leads to the following integral equation of $\Psi$,

$$(6) \qquad \Psi(E_a, E_b) = 2\pi F(\boldsymbol{r}, E_a - E_b)\Phi_a + \int_{-\infty}^{\infty} \mathrm{d}p F(\boldsymbol{r}, p)(E_b + p - H_0)^{-1} \Psi(E_a, E_b + p) .$$

Eq. (6) represents a simple form of time-dependent perturbation theory which includes as a special case the « sudden perturbation » where we have

$$H_1(\boldsymbol{r}, t) = \begin{cases} 0; & t < 0 \text{ and } \tau < t, \\ H_1(\boldsymbol{r}); & 0 \leqslant t \leqslant \tau, \end{cases}$$

and therefore

$$F(\boldsymbol{r}, p) = -\frac{1}{2\pi} \frac{\exp[-ipt] - 1}{ip} H_1(\boldsymbol{r}) ,$$

which leads at once to the first order result in Dirac's theory.

If one considers next a perturbation finite and continuous or piecewise continuous in time and use the artifice of a damping factor $\exp[-\varepsilon(t)]$, Eq. (1) becomes

$$H_1(t) = \exp[iH_0 t] H_1(\boldsymbol{r}, t) \exp[-\varepsilon(t)] \exp[-iH_0 t] .$$

One obtains after some calculation,

$$(7)\quad \Psi(E_a, E_b) = 2\pi F(\boldsymbol{r}, E_a - E_b)\Phi_a + \int_{-\infty}^{\infty} \mathrm{d}p\, F(\boldsymbol{r}, p)(E_b - H_0 + p + i\varepsilon)^{-1}\Psi(E_a, E_b + p) +$$

$$+ 2\varepsilon \int_{-\infty}^{\infty} \mathrm{d}p\, F(\boldsymbol{r}, p)[\varepsilon^2 + (E_b + p - H_0)^2]^{-1} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}p_1 \cdots \mathrm{d}p_n \cdot$$

$$\cdot \left[(ni\varepsilon + E_a - H_0 - p_1 - \cdots - p_n)^{-1} - ((n+1)i\varepsilon + E_a - E_b - p - p_1 - \cdots - p_n)^{-1}\right] F(\boldsymbol{r}, p_1) \cdot$$

$$\cdot ((n-1)i\varepsilon + E_a - H_0 - p_2 - \cdots - p_n)^{-1} F(p_2, \boldsymbol{r}) \cdots (i\varepsilon + E_a - H_0 - p_n)^{-1} F(\boldsymbol{r}, p_n)\Phi_a .$$

The limiting process $\varepsilon \to 0$ is to be performed after the transition matrix element is calculated.

If $H_1$ is independent of time and $H_0$ has a simple continuous spectrum, then $F(\boldsymbol{r}, E_a - E_b)$ becomes $H_1\delta(E_a - E_b)$ and it can be shown that Eq. (7) reduces to the integral equation [1]

$$\Psi_a = \Phi_a + (E_a - H_0 + i\varepsilon)^{-1}\Psi_a .$$

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# Charge Independence and Pion Scattering.

A. Stanghellini
*CERN - Geneva* (*)

(ricevuto il 5 Agosto 1958)

The matrix elements of pion nucleon scattering for different charge states are related by charge independence. If one admits the existence of a charge triplet pion (1), only the two amplitudes $T_{\frac{3}{2}}$ and $T_{\frac{1}{2}}$ are necessary to describe the scattering from nucleon. For reference, we write down the well-known relation between the experimental processes and those amplitudes:

$$(1)\qquad \begin{cases} (\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p}) & T+(\theta,\omega) = T_{\frac{3}{2}}(\theta,\omega)\,, \\ (\pi^- + \mathrm{p} \to \pi^- + \mathrm{p}) & T-(\theta,\omega) = \tfrac{1}{3}T_{\frac{3}{2}}(\theta,\omega) + \tfrac{2}{3}T_{\frac{1}{2}}(\theta,\omega)\,, \\ (\pi^- + \mathrm{p} \to \pi^0 + \mathrm{n}) & T_0(\theta,\omega) = \dfrac{\sqrt{2}}{3}T_{\frac{3}{2}}(\theta,\omega) - \dfrac{\sqrt{2}}{3}T_{\frac{1}{2}}(\theta,\omega)\,. \end{cases}$$

The proof that the charge independence hypothesis is a good one is achieved if the amplitudes $T_{\frac{1}{2}}$ and $T_{\frac{3}{2}}$ are the same in the three different processes (1).

However, for different reasons it is very difficult to have a quantitative proof:

*a*) The different experimental accuracy for the three processes.

*b*) Ambiguities in the phase-shifts analysis.

*c*) The errors to be assigned to each phase-shift.

The present experimental situation suggests that the charge independence hypothesis is a good one. The purpose of this paper is to see whether it is possible to give a quantitative ground to this feeling, in a direct way, without passing through the phase-shift analysis.

We list some relations between cross-sections and angular distributions for the three processes (1), which are necessary consequences of charge independence:

$$(2)\qquad 3\sigma_T^- = (\sigma_+ + 2\sigma_{\frac{1}{2}})\,,$$

$$(3)\qquad 3(\mathrm{d}\sigma^- + \mathrm{d}\sigma^0) = \mathrm{d}\sigma^+ + 2\,\mathrm{d}\sigma_{\frac{1}{2}}\,.$$

(*) On leave of absence from the Istituto di Fisica dell'Università, Bologna.

(1) If a scalar $\pi_0'$ exists, one has to add an amplitude $T'(\theta,\omega)$ to $T_0(\theta,\omega)$.

These two relations give the following inequalities (2)

$$(4) \qquad \begin{cases} 3\sigma_T^- - \sigma_+ \geqslant 0\,, \\ 3(a_- + a_0) - a_+ \geqslant 0\,, \end{cases}$$

due to the fact that $\sigma_1$ (total cross-section) and $a_1$ (the constant term of the angular distribution: $a_1 = \mathrm{d}\sigma_1(90°)$) are positive quantities. These relations are fulfilled by the experimental data, but they are not sufficiently stringent conditions for our scope.

A more interesting relation is

$$(5) \qquad T_0(\theta, \omega) = \frac{1}{\sqrt{2}}\left(T_+(\theta, \omega) - T_-(\theta, \omega)\right).$$

We take this for $\theta = 0$ and square. We have

$$(6) \qquad 2\,\mathrm{d}\sigma_0(0) = (|\operatorname{Re} T_+(0) - \operatorname{Re} T_-(0)|)^2 + \left(\frac{\eta}{4\pi}\right)^2 (\sigma_+ - \sigma_-)^2\,,$$

$$\operatorname{Re} T_\pm(0) = \sqrt{\mathrm{d}\sigma_\pm(0) - \left(\frac{\eta\sigma_\pm}{4\pi}\right)^2}\,.$$

Table I gives the experimental values for the two sides of (6).

TABLE I. — *Cross-sections measured in* $\eta_0^2 = 20$ mb *units.*

| $\pi$ kinetic energy MeV | $a_L$ | $a_R$ | $\dfrac{\Delta(a_L - a_R)}{a_L}$ |
|---|---|---|---|
| 150 (*) | $0.38 \pm 0.02$ | $0.36 \pm 0.04$ | 12% |
| 220 (*) | $0.64 \pm 0.04$ | $0.67 \pm 0.12$ | 18% |
| 240 (+) | $0.66 \pm 0.05$ | $0.60 \pm 0.07$ | 12% |
| 270 (+) | $0.52 \pm 0.04$ | $0.45 \pm 0.07$ | 15% |
| 307 (+) | $0.48 \pm 0.04$ | $0.41 \pm 0.07$ | 17% |

(*) J. ASHKIN, J. P. BLASER, F. FEINER and M. O. STERN: *CERN Symposium* (1956), p. 225.

(+) A. I. MUHIN, E. B. OZEROV, B. M. PONTECORVO, E. L. GRIGORIEV and N. A. MITIN: *CERN Symposium* (1956), p. 201; S. M. KORENČENKO and V. G. ZINOV: preprint and private communication.

The 0 angle extrapolation is made from the differential cross-section of the form $(a + b\cos\theta + c\cos^2\theta)$, *i.e.* $S$- and $P$-waves are present. We see that the two sides are equal within 10%.

The experimental error is of the same order of magnitude. This means that the relation (6) is fulfilled and our knowledge is not better than 15%.

(2) There are also the so-called « triangle inequalities »

$$\sqrt{2}\sqrt{\mathrm{d}\sigma_0} - \sqrt{\mathrm{d}\sigma_-} \leqslant \sqrt{\mathrm{d}\sigma_+} \leqslant \sqrt{2}\sqrt{\mathrm{d}\sigma_0} + \sqrt{\mathrm{d}\sigma_-}\,,$$

used by SALZMAN (preprint) with the same scope.

Another problem connected with charge independence is the value of the $T_{\frac{1}{2}}$ amplitude and its accuracy. The energy region considered above is not the best for a direct inspection. In fact, it is also the region where the $T_{\frac{3}{2}}$ amplitude is so large that practically all the experimental information comes from it.

Nevertheless, it is interesting to see to what extent this amplitude is known, because several speculations were made (linearity of $S$-waves, sign of $P$-waves, behaviour of $D$-waves).

From (2) and (3) we can obtain the total cross-sections and the coefficient of the angular distributions for this amplitude. We have:

TABLE II.

| $\pi$ kinetic energy MeV | $\sigma_1$ (mb) | $a_1$ | $b_1$ | $c_1$ |
|---|---|---|---|---|
| 150 | 7.5±7 | 0.07±0.3 | −0.18±0.23 | 0 ±0.5 |
| 170 | 1 ±7 | −0.30±0.23 | −0.04±0.25 | 0.1±0.5 |
| 220 | 1 ±6 | 0.25±0.30 | −0.21±0.25 | 1.5±0.9 |
| 240 | 5 ±5 | 0.90±0.35 | 0.55±0.60 | −0.5±1.5 |
| 270 | 14 ±5 | 1.25±0.30 | 0.35±0.50 | 2.1±1.0 |
| 307 | 15 ±4 | 1.14±0.30 | 0.50±0.50 | 0.2±1.0 |

We see that there is very little outside the errors. One might use the Ashkin and Vosko method, or any other methods [3], but it is hopeless with this angular distribution.

From experiments one can get further information relative to $T_1$, *i.e.* the interference between $T_3$ and $T_1$ amplitude; $T_3$ is large, so we might find somethind outside the errors. From (1)

$$\mathrm{Re}\,(T_1^+ \cdot T_3) = \tfrac{1}{4}(3|T_-|^2 - 6|T_0|^2 + |T_3|^2) \quad \text{or} \quad a_{\mathrm{int}} = \tfrac{1}{4}(3a_- - 6a_0 + a_+) \quad \text{etc.}.$$

Table III gives the experimental values.

TABLE III.

| $\pi$ kinetic energy (MeV) | $4a_{\mathrm{int}}$ | $4b_{\mathrm{int}}$ | $4c_{\mathrm{int}}$ |
|---|---|---|---|
| 150 | 1.3 ± 0.5 | 7.0 ± 0.6 | 0.2 ± 1.4 |
| 170 | 1.9 ± 0.6 | 5.1 ± 0.7 | 2.3 ± 1.7 |
| 220 | 0.8 ± 1.2 | −0.6 ± 1.0 | −3.3 ± 2.7 |
| 240 | −1.3 ± 1.6 | −2.1 ± 2.0 | −0.5 ± 4.5 |
| 270 | −1.7 ± 1.2 | −4.7 ± 1.3 | −2.1 ± 3.5 |
| 307 | −0.7 ± 1 | −4.3 ± 1.5 | −1.5 ± 3.5 |

[3] E. CLEMENTEL and C. VILLI: *Suppl. Nuovo Cimento*. They have used these angular distributions to get the phase-shifts.

Due to the fact that $T_1^2$ is small, we make the small phase-shifts approximation (*i.e.* $T_1$ real).

$$\mathrm{Re}\,(T_1^+ T_3) = T_1\,\mathrm{Re}\,T_3\,,$$

$$a_{\mathrm{int}} = \lambda^2\{\alpha_1\,\mathrm{Re}\,A_3 + (\alpha_{13} - \alpha_{11})\,\mathrm{Re}\,C_3\}\,,$$

$$b_{\mathrm{int}} = \lambda^2\{\alpha_1\,\mathrm{Re}\,B_3 + (2\alpha_{13} + \alpha_{11})\,\mathrm{Re}\,A_3\}\,,$$

$$c_{\mathrm{int}} = \lambda^2\{(2\alpha_{13} + \alpha_{11})\,\mathrm{Re}\,B_3 - (\alpha_{13} - \alpha_{11})\,\mathrm{Re}\,C_3\}\,,$$

$$\mathrm{Re}\,A_3 = \sin\alpha_3\,\cos\alpha_3\,,$$

$$\mathrm{Re}\,B_3 = 2\sin\alpha_{33}\cos\alpha_{33} + \sin\alpha_{31}\cos\alpha_{31}\,,$$

$$\mathrm{Re}\,C_3 = 2\sin\alpha_{33}\cos\alpha_{33} - \sin\alpha_{31}\cos\alpha_{31}\,.$$

The combinations

$$a_{\mathrm{int}} + b_{\mathrm{int}} + c_{\mathrm{int}} = \frac{1}{\eta_l}(\alpha_1 + 2\alpha_{13} + \alpha_{11})\,\mathrm{Re}\,T_3(0)\,,$$

$$a_{\mathrm{int}} - b_{\mathrm{int}} + c_{\mathrm{int}} = \frac{1}{\eta}(\alpha_1 - 2\alpha_{13} - \alpha_{11})(2\alpha_{3/\eta} - \mathrm{Re}\,T_3(0))$$

are useful, in order to get $\alpha_1$ and $2\alpha_{13} + \alpha_{11}$.

We have

| | $\alpha_1$ | $2\alpha_{13} + \alpha_{11}$ | $\alpha_{13} - \alpha_{11}$ [4] |
|---|---|---|---|
| 150 | $\sim 13^\circ \pm 3^\circ$ | $\sim 5^\circ \pm 3^\circ$ | $\sim 8^\circ \pm 5^\circ$ |
| 270 | $\sim 9^\circ \pm 5^\circ$ | $\sim 8^\circ \pm 5^\circ$ | |
| 307 | $\sim 11^\circ \pm 7^\circ$ | | |

The data near the resonance cannot be used since $T_3$ changes very rapidly. We can make the following conclusions.

i) Only $\alpha_1$ is known up to 307 MeV. It is positive. It is very difficult to derive a conclusion on its behaviour on energy.

ii) The combinations $2\alpha_{13} + \alpha_{11}$ and $\alpha_{13} - \alpha_{11}$ seem to be positive. All the data are consistent with

$$\alpha_{13} = \alpha_{11} = 0\,.$$

The analysis of the 307 MeV experiments needs some comments. One should expect $D$-waves at this energy. We think that $S$ and $P$ analysis is a good approximation

(4) We have put in this table only the phase-shifts outside the errors.

in view of the fact that experiments have not been carried out on small and large angles. As inelastic processes by negative $\pi^0$ seem to be large (of the order of 10% of elastic cross-section), real phase-shifts analysis is of no value.

We apologise that this is not a complete analysis of phase-shifts, but only a qualitative one in order to give the best information for the electronic computer.

* * *

I would like to thank Prof. J. ASHKIN for his assistance in offering helpful criticisms and for his time spent in discussing this subject.

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# Operation of a Bubble Chamber Filled with «High $Z$» Mixtures.

L. Bertanza, P. Franzini, I. Mannelli and V. Silvestrini

*Istituto di Fisica dell'Università - Pisa*
*Istituto Nazionale di Fisica Nucleare - Sezione di Pisa*

(ricevuto il 12 Agosto 1958)

It was pointed out by various authors that the use in a bubble chamber of a liquid of high atomic number and density may offer many actractive possibilities, owing to high $\gamma$-conversion efficiency, stopping power and so on. Multiple Coulomb scattering may rule out the possibility of measuring track curvatures in magnetic field, but, on the other hand, it may yeld some further information.

Actually a number of «high $Z$» bubble chambers have been built [1-2], but it seemed worthwhile to investigate the possibility of using liquids which present more favorable features, in respect of cost and availability, chemical stability and corrosion properties, temperature and pressure of operation, etc. Moreover, we have searched for liquids showing these features and containing a high partial density of free hydrogen.

To this aim, we have used an experimental stainless steel chamber, cilindrical in shape (6 cm i.d., 3 cm deep). The chamber is operated by means of compressed air acting on a membrane of neoprene faced with a layer of 0.2 mm mylar and supported by a perforated plate.

A «venetian blind» provides a dark field illumination of the chamber which is photographed by a commercial camera.

In order to work at or near room temperature, the chamber was operated as a «gas bubble chamber» of the kind first realized by P. E. Argan and A. Gigli [3] and independently proposed by G. A. Askar'ian [4]. Successful operation was obtained with methyl-

Fig. 1.

[1] J. L. Brown, D. A. Glaser and M. L. Perl: *Phys. Rev.*, **102**, 586 (1956).

[2] E. D. Alyea jr., L. R. Gallagher, J. H. Mullins and J. M. Teem: *Nuovo Cimento*, **6**, 1480 (1957).

[3] P. E. Argan and A. Gigli: *Nuovo Cimento*, **3**, 1171 (1956).

[4] G. A. Askar'ian: *Sov. Phys. Journ. Exp. Theor. Phys.*, **1**, 571 (1955).

TABLE I.

| System | Fractional weight of $CH_3I$ | Equil. press. (atm) | Final press. (atm) | Temp. (°C) | Density ($g/cm^3$) (expanded state) | Hydrogen density ($g/cm^3$) | Radiat. lenght (cm) | Fig. |
|---|---|---|---|---|---|---|---|---|
| $CH_3I+C_3H_8+C_2H_6$ | 35% | 30.5 | 10 | 28 (room) | .65 | .085 | 27.3 | 1 |
| $CH_3I+C_3H_8+C_2H_6$ | 70% | 31.5 | 12 | 28.5(room) | 1.05 | .074 | 11.4 | 2 |

iodide-ethane and methyl-iodide-carbon dioxide systems, with (and without) a small quantity of propane, which increases the solubility of the gas.

Fig. 2.

Varying the percentage of $CH_3I$, it is possible to vary largely the radiation lenght of the liquid (till less than 10 cm), the mixture remaining completely colourless for any lenght of time we have operated. In Table I the operating conditions are given together with the other interesting data relative to the two reported pictures (Figs. 1 and 2), showing tracks of Compton electrons from the $\gamma$'s of a radioactive source. The bubble density is estimated to be about 20 bubbles/cm in both cases.

* * *

We wish to thank Prof. A. GIGLI for useful suggestions.

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# Radiative Corrections to the $\pi \to e + \nu$ Decay (*).

H. W. Lewis (+)

*University of California, Los Alamos Scientific Laboratory - Los Alamos, New Mexico*

(ricevuto il 19 Agosto 1958)

It has recently been suggested (1) that the surprisingly low value of the ratio of the $\pi \to e + \nu$ process to the $\pi \to \mu + \nu$ process may have a simple explanation. Simple perturbation theory applied to the weak-coupling vertex, using the $V$-$A$ interaction, leads to the result that the ratio is proportional to $(m/\mu)^2$, where $m$ is the mass of the electron and $\mu$ is the mass of the $\mu$-meson. The suggestion is that the masses that appear in this ratio ought to be the « bare » masses of the electron and meson, and that the ratio of these latter may be much smaller than the ratio of the « observed » masses. It is the purpose of this note to explore the physical content of such an argument.

First, we may note that the comment is frequently made that the bare mass of the electron is unobservable and cannot appear in any physical process. This statement is certainly incorrect in a convergent theory, in which the bare mass and charge of the particles involved are just as characteristic for the high energy processes in which it participates as are renormalized mass and charge for the low energy processes (2). Whether this is also true for divergent theories is a moot point.

Thus, the argument mentioned above may be stated in a slightly different form. One may say that the meson decay takes place in such a short time that the electron (or $\mu$-meson) doesn't have any time to get dressed, that is, to put out its electromagnetic cloud. In the language of perturbation theory, the electron line connecting to the decay vertex is always a bare electron line, whence the matrix element contains the bare mass. Only after the electron has escaped from the decay center does it establish the surrounding field which is responsible for the electromagnetic part of its mass. (We ignore the quantitative question of the fraction of the electromagnetic mass of the electron for which this argument is valid; it is probably small).

We wish here only to point out that there is another, purely electromagnetic,

(*) Work performed under the auspices of the U. S. Atomic Energy Commission.

(+) Permanent address: University of Wisconsin, Madison, Wisconsin.

(1) R. Gatto and M. A. Ruderman: *Nuovo Cimento*, **8**, 775 (1958).

(2) M. Gell-Mann and F. E. Low: *Phys. Rev.*, **95**, 1300 (1954); L. D. Landau: in *Niels Bohr and the Development of Physics* (London, 1955).

process to which one can apply a similar argument, in which case it is easy to show that other electromagnetic processes exactly cancel the effects one is considering here.

If we scatter a Dirac electron from a static weak Coulomb potential, it is trivial to show that the differential scattering cross section is given by

$$(1) \qquad \frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_R \left[1 - \frac{v^2}{c^2}\sin^2\left(\frac{\vartheta}{2}\right)\right],$$

where $(d\sigma/d\Omega)_R$ is the Rutherford cross section, $v$ is the velocity of the electron, and $\vartheta$ is the angle of scattering. Thus, the cross section for back-scattering is multiplied by $(1 - v^2/c^2)$, and is exactly zero if $v = c$ (which is the case if the rest mass of the electron is zero). We know that (1) is correct, using the *observed* mass of the electron, provided that the momentum transfer in the scattering is not too great.

Now consider the effect of starting with (1) for the bare electron and then taking into account the radiative corrections to the mass. Again we can argue that the insertion of an electron self-energy part into either the incoming or the outgoing electron line will have no effect on the vanishing of the cross section for back-scattering, so that the argument appears to be going as it did before. However, we have so far neglected the vertex corrections to the scattering, and it is easy to see that they give non-vanishing contributions to the process, even for zero bare mass of the electron. Indeed, we know that the combined effect of the vertex and self-energy corrections is just to lead to (1) with the observed mass of the electron. This is the substance of the proof of the renormalizability of quantum electrodynamics.

If the electron had zero bare mass, therefore, we see that an argument neglecting the vertex corrections would lead to a vanishing cross section for back-scattering from a Coulomb field, whereas when the vertex corrections are taken into account, we are led to the correct, well-known, formula (1).

This discussion cannot be taken over unchanged to the case of the $\pi \to e + \nu$ decay, because the vertex in that case is presumably a very complicated object, involving mesons and nucleons. However, it is clear that one must be careful in applying the electrodynamic corrections to the out-going charged particle only.

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# Quantum Numbers for a System of Nucleons and Antinucleons.

A. Gamba and E. C. G. Sudarshan (*)

*University of Rochester, Departement of Physics - Rochester, N. Y.*

(ricevuto il 26 Agosto 1958)

The most general hermitian operator on the charge state of a nucleon can be expressed as a linear combination of the unit matrix and the three two-by-two isotopic spin matrices $\tau_1$, $\tau_2$, $\tau_3$: It is also well-known that the charge state of a many-nucleon system is completely defined (1) by two quantum numbers $I$ and $I_3$ which are conserved by virtue of the charge-independence of nuclear forces. Electromagnetic interactions which destroy the conservation of $I$ (but not $I_3$) represent only minor corrections to the conservation of the « total » isotopic spin $I$ and the classification of a many-nucleon system using $I$ is one which displays the constants of motion.

In this case the isotopic spin $I$ may be defined in one of several equivalent ways, as describing the invariance of the dynamical laws under the following groups of transformations:

*a*) Unitary transformations in the (2-dimensional) « space » spanned by the charge states of the nucleon.

*b*) Real rotations in a 3-dimensional « isotopic spin space ».

*c*) Permutations of the charge coordinates of the individual particles of the many-nucleon system. (Because of the Pauli principle, this is equivalent to permutations of the space and spin coordinates of the nucleons).

When both nucleons and antinucleons are present, since the charge states are now increased to four (p, n, $\bar{\text{n}}$, $\bar{\text{p}}$) the most general hermitian matrix is a linear combination of 16 Dirac matrices. It is then clear that $I$ and $I_3$ no longer suffice to classify the charge states of the system. Whereas $I$ still labels the irreducible representations of group *b*), we need a total of three quantum numbers to label the irreducible representations of the groups in *a*) (2) and/or *c*). The problem is mathematically the same as considered by Wigner in his supermultiplet theory (3). Unfortunately his

(*) On leave from Tata Institute of Fundamental Research, Bombay, India.

(1) Apart from an irrelevant permutation degeneracy, which is removed as soon as the exclusion principle is taken into account.

(2) Where group (*a*) is now the group of unitary transformations in the (4-dimensional) space spanned by the charge states (proton, neutron, antineutron, antiproton).

(3) E. Wigner: *Phys. Rev.*, **51**, 106 (1937): E. Wigner and E. Feenberg: *Rep. Prog. Phys.*, **8**, 274 (1941).

$T$, $S$, $Y$ (sometimes denoted by $P$, $P'$, $P''$) bear no simple relation to the total isotopic spin $I$. A more satisfactory choice could be a set containing the total isotopic spin $I$, the « total » baryon number $K$ [4] and a suitably chosen third number $H$ which could be constructed explicitly. To specify a definite component of this irreducible representation (*i.e.*, the complete wave function) in charge space one needs, in general, three « third components » $I_3$, $K_3$ and $H_3$ (cfr. reference [3]).

For a system of one nucleon and one antinucleon, this full set of labels ($I$, $K$, $H$) is redundant; and the classification (as well as $K$ and $H$ themselves) is completely defined in terms of $I$ and the multiplicative quantum numbers $G$ [5] and these are constants of motion. This nucleon-antinucleon system is however an exception like the deuteron in supermultiplet theory [6]. In the general case it can be shown that *there are no additive quantum numbers for the nucleon-antinucleon-pion system in addition to $I$, $I_3$ and $K_3$ which are strictly conserved* (*if one accepts the usual charge-independent* PS(PS) *or* PS(PV) *interaction*). This circumstance is due to the particular dynamics of the system. It is of interest to examine the possibility of approximate conservation of some of the additional quantum numbers related to the charge degrees of freedom at least in certain restricted domains of energy and/or for specific reactions.

Regarding an experimental test of such a possibility, the simplest non-trivial system must contain at least 3 particles with both nucleons and antinucleons. Such a process is the non-annihilation scattering of an antinucleon by deuterons. Consider the reactions

(A) $\bar{n}+d \to \bar{n}+p+n$,

(B) $\bar{n}+d \to \bar{p}+p+p$,

(A′) $\bar{p}+d \to \bar{p}+p+n$,

(B′) $\bar{p}+d \to \bar{n}+n+n$.

The final states are classified in Table I.

TABLE I.

| Quantum numbers for final states | | | Statistical Weight | R |
|---|---|---|---|---|
| $I$ | $K$ | Overall symmetry of charge space | | |
| $\frac{1}{2}$ | $\frac{1}{2}$ | symmetric | 1 | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | intermediate symmetry; Young pattern [21] | 2 | 2 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | antisymmetric | 1 | $\infty$ |
| $\frac{1}{2}$ | $\frac{3}{2}$ | intermediate symmetry; Young pattern [21] | 2 | 2 |

Charge-independence in the usual sense only requires that the ratio $R$ of the differential cross-sections

$$R = \frac{\sigma(A)}{\sigma(B)} = \frac{\sigma(A')}{\sigma(B')},$$

be restricted by

$$\tfrac{1}{2} \leqslant R \leqslant \infty .$$

---

(4) $K$ is related to the baryon number $K_3$ (number of nucleons minus antinucleons) as $T$ is related to $T_3$. See B. TOUSCHEK: *Nuovo. Cimento*, **8**, 181 (1958); O. HARA and Y. FUJII: *Progr. Theor. Phys.*, **17**, 313, 819, 820, 822 (1957).

(5) T. D. LEE and C. N. YANG: *Nuovo Cimento*, **3**, 749 (1956): the same number has been introduced with somewhat different notation by D. AMATI and B. VITALE: *Nuovo Cimento*, **2**, 719 (1955) and C. J. GOEBEL: *Phys. Rev.*, **103**, 258 (1956).

(6) The Bartlett and Heisenberg forces, which in general destroy the validity of the quantum numbers which specify a supermultiplet, do not destroy the validity of the supermultiplet picture in the case of the deuteron.

If the final state belongs to a definite irreducible representation one obtains the values listed in the last column of the table. An experimental value $R \geqslant 2$ would not be very significant; however, a value $2 \geqslant R \geqslant \frac{1}{2}$ should further enable us to infer a resonance in the $(\frac{1}{2}, \frac{1}{2}$, symmetric) state.

The point we wish to emphasize is that even within the framework of charge-independent interactions not all final states are necessarily equivalent and the additional quantum numbers labelling these states may be sufficiently well conserved, at least for a certain range of processes. That the existence of new selection rules related to conservation of quantum numbers like strangeness *necessarily* requires the introduction of additional internal degrees of freedom seems to us unwarranted. There is wealth of internal symmetry labels for many-particle systems which could be exploited in a suitable composite model of strange particles. We believe that this problem has not been sufficiently investigated to provide a definite verdict on the success of such a program.

* * *

We wish to acknowledge useful discussions with C. J. Goebel, M. F. Kaplon, R. E. Marshak, S. Okubo and A. Salam.

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# On the Applicability of $^4$He Gas Scintillators as Analysers of Neutron Energy and Polarization.

C. Rubbia and M. Toller

*Istituto di Fisica dell'Università - Pisa*
*Istituto Nazionale di Fisica Nucleare - Sezione di Pisa*

(ricevuto il 29 Agosto 1958)

The technique of $^4$He gas scintillators, developed by various authors ([1,2]), has been recently applied by Pasma ([3]) to measure the polarization of neutrons emitted in D+D and D+T reactions at deuteron energies up to 500 keV. In these experiments the neutron polarization was deduced from left-right asymmetries in the elastic scattering on $^4$He, by means of coincidences between the helium recoil nuclei (detected by the gas scintillation) and the scattered neutrons. These asymmetries are quite large and little dependent on the neutron energy in the whole range (2÷20) MeV ([4]).

Limitations to a wider applicability of this technique arise essentially from the requirement of a high intensity for the source producing the neutrons to be analyzed, owing to the low density of the He scatterer. It seemed therefore worth-while to investigate the scintillating properties of He under conditions of high pressure and low temperature. For this purpose we have built a stainless steel counter (see Fig. 1) with teflon gaskets and with a thick glass window. The inner surface has been coated with a MgO film, and about 50μg/cm$^2$ of quaterphenyl were evaporated under vacuum on it, as well as on the glass window ([1,2]). A $^{210}$Po α source was mounted inside the counter, viewed by a phototube EMI 6097F. The counter, filled with spectroscopically pure helium, was tested in the pressure interval from 20 to 100 atm and at the temperatures 20 °C, 0 °C and — 60 °C. No appreciable variation of the pulse height was found within the estimated accuracy of 10%.

At room temperature (about 25 °C) a decrease of about 50% was noticed during a week, while at 0 °C no change in pulse height was found in the same time interval. As already known ([1]), this effect is probably due to gas poisoning by quaterphenyl vapours. We found, however, that the counter recovers its old scintillating properties by cooling it for a short time at solid $CO_2$ temperature.

The shape of the pulses was investigated using a Tektronix type 517 oscillo-

([1]) R. A. Nobles: *Rev. Sci. Instr.*, **27**, 280 (1956).

([2]) A. Sayres and C. S. Wu: *Rev. Sci. Instr.*, **28**, 758 (1957).

([3]) P. G. Pasma: *Nucl. Phys.*, **6**, 141 (1958) also for other references.

([4]) I. I. Levnitov. A. V. Miller and V. N. Šamšev: *Nucl. Phys.*, **3**, 221 (1957).

scope. Rise and decay times were both about 15 ns, essentially due to the time spread of the photomultiplier. The number of photons collected by the photocathode has been estimated about 1400 per MeV lost by the particle in the gas.

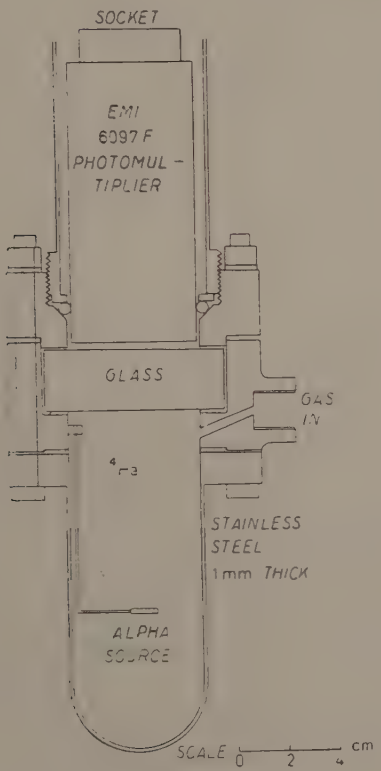


Fig. 1.

The calculated over-all efficiency of a neutron polarimeter ([3]) making use of the He counter previously described, is several times $10^{-4}$. This figure is obtained for a geometry such that a left-right asymmetry of $\sim 0.8$ would be recorded with completely polarized neutrons of 14 MeV. For such a geometry, the scattering angle is still sufficiently determined to make it possible to measure, from the pulse height of the He counter, the neutron energy with $\sim \pm 10\%$ accuracy.

The possibility of applying this technique to the investigation of elementary processes involving $\pi$-mesons should not be overlooked. In fact, with the efficiency given above, experiments on polarized neutrons emitted in reactions of the type $\gamma + p \rightarrow n + \pi^+$ or $\pi^- + p \rightarrow$ $\rightarrow n + \pi^0$ turn out to be feasible up to primary energies of several hundred MeV.

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# LIBRI RICEVUTI E RECENSIONI

BRUNO DE FINETTI – *Matematica logico-intuitiva*, ed. Cremonese, Roma, 1957.

È piuttosto diffusa tra i profani l'idea che la matematica sia una disciplina arida e conchiusa, buona al più come passivo strumento nelle mani di pochi specialisti, ma alla quale possono interessarsi *ex professo* solo persone di poca fantasia, meno sensibili a quella che è l'apertura senza limiti, l'inesausta fecondità della vita nella sua pienezza. Tra gli stessi matematici v'è chi, nell'insegnamento, pone l'accento sul rigore formale sino a dare l'impressione che in esso si esaurisca la loro scienza, come pure chi, al contrario, valorizza l'intuizione in modo del tutto unilaterale, quasi che il rigore logico fosse una superflua pedanteria.

A tutti coloro che così pensano sarebbe preziosa la lettura del capitolo introduttivo del presente libro, per renderli avvertiti della vera natura della nostra scienza e dello scopo profondo del suo insegnamento.

È vero che, da un lato, la matematica non si differenzia concettualmente dalla logica se non per il suo carattere specializzato; questa, come quella, insegna a *ragionare*, ossia a formulare con chiarezza e precisione i propri giudizi ed a scoprirne la reciproca interdipendenza (in particolare, a dedurre una proposizione da un'altra); è questo il lato del rigore formale, che richiede, nel matematico come nel logico, *senso critico* diretto a preservare dagli errori, e *senso d'astrazione* capace di cogliere l'unità organica che si può celare dietro una apparente molteplicità (« economia malintesa è quella che fa cercare di evitare ogni considerazione che non sia materialmente indispensabile per giungere alla conclusione; economia razionale è quella consistente nell'inquadrare il problema particolare in quello più generale che consenta la visione più semplice ed approfondità della questione », p. 13). L'insegnamento deve tendere soprattutto a destare ed educare tali capacità.

Ora, l'esigenza del rigore formale si soddisfa mercè l'uso di procedimenti codificati (« algoritmi » matematici, o, più generalmente, logici) che, quasi macchine per pensare, rappresentano un aiuto potente e spesso insostituibile del pensiero vivente; bisogna guardarsi, però, dall'usare tali algoritmi meccanicamente, perchè sarebbe « la stessa aberrazione di chi, per sfruttare al massimo l'agevolazione data dall'automobile per visitare diverse regioni, le percorresse a massima velocità senza staccare lo sguardo dal tachimetro » (p. 27): studiando un problema bisogna invece tener presente che « molti elementi utili per comprendere ... i suoi vari aspetti, per intuire nuove relazioni attraverso insospettate associazioni d'idee, e insomma per mantenere allo stato vivo, fluido, fecondo le proprie

conoscenze, scaturiscono dall'associare incessantemente ad ogni passaggio formale la sua interpretazione intuitiva e ad ogni suggerimento dell'intuizione, la ricerca di un progresso formale. A queste condizioni ... l'algoritmo diventa, per così dire, una forma di pensiero tutt'uno con la visione concreta e intuitiva » (p. 28).

Veniamo così all'altro aspetto della matematica, quello fantastico-intuitivo, non meno importante ed anzi complementare del primo, perchè accanto alla logica anche la fantasia e l'intuizione sono mezzi indispensabili di conoscenza: se l'intuizione coglie la realtà attuale, solo la fantasia ha la capacità di oltrepassarla per concepire anche ciò che non è attualmente reale, mentre la logica le impedisce di cadere nell'incoerente e nell'assurdo. Si può dire, quindi, che la matematica è viva e feconda in quanto alimentata da una fantasia creatrice sorretta dall'intuizione e disciplinata dalla logica: la fantasia creatrice, appunto, che al di là dell'*ovvio* e del *consueto* indica il *possibile* ed i mezzi per *realizzarlo*. L'insegnamento deve rassodare l'intuizione e stimolare la fantasia, quindi, pena la sterilità.

Queste considerazioni, in parte riassunte o parafrasate dal primo capitolo del libro in esame, ne spiegano il singolare titolo *Matematica logico-intuitiva*, e ne illustrano la concezione ed il metodo. In questo libro, coerentemente, gli argomenti tradizionali d'un corso di Matematiche Generali (Aritmetica, Algebra lineare, Geometria analitica, Teoria dei limiti, Funzioni, Calcolo differenziale ed integrale) vengono presentati come diverse articolazioni d'un tutto organico, la conoscenza matematica, e questa come una specificazione della più generale conoscenza umana; la trattazione è tale da prevenire (per quanto possibile) che nella mente del lettore si formino paratie stagne, e favorisce al massimo il ricambio e l'apertura delle idee, anche con esempi illuminanti e con esercizi suggestivi, tratti da tutti i rami della matematica ed anche dalla vita comune, in modo proprio insolito per testi del genere.

In conclusione, si tratta di un libro di grande interesse didattico, ben oltre la portata d'un corso di Matematiche Generali ad uso degli studenti di Economia e Commercio, e merita d'essere meditato da chiunque professi l'insegnamento della matematica; è augurabile che l'autore abbia modo di applicare queste sue idee in un trattato di carattere più elevato, rivolto a studenti di Matematica e Fisica, ove possa svolgere in modo armonico e compiuto ciò che in questa sede è per forza di cose appena accennato.

Veniamo ora ad esporre in particolare il contenuto del libro. Nel capitolo introduttivo, oltre alle osservazioni esposte in principio, si dànno gli elementi della logica proposizionale bivalente (essenziale per una fondazione chiara del Calcolo delle Probabilità, ed utilissima in tutto lo svolgimento del corso) ed i concetti fondamentali della matematica, come quelli di insieme, corrispondenza, operazione e trasformazione (funzione). A formare la prima parte del corso (« Matematiche complementari »), seguono altri cinque capitoli, rispettivamente dedicati: i) agli insiemi finiti, e quindi all'aritmetica dei numeri interi (introdotti come numeri cardinali, ciò che consente un cenno sulle potenze del numerabile e del continuo) ed all'analisi combinatoria (notevolmente sviluppata, come conviene come premessa al Calcolo delle Probabilità); ii) alle classi di grandezze omogenee, e quindi all'aritmetica dei numeri reali ed alle funzioni « elementari » nel campo reale; iii) alle similitudini piane, e quindi all'aritmetica dei numeri complessi, ed alle funzioni elementari nel campo complesso, con un cenno sulle funzioni analitiche ed alla loro classificazione; iv) ai sistemi lineari, e quindi ai vettori, alle matrici, all'algebra lineare (da osservare la definizione *descrittiva* di determinante); v) agli elementi di geometria analitica piana e spaziale, con la

classificazione dei concetti geometrici secondo il gruppo rispetto a cui sono invarianti.

La seconda parte (« Calcolo differenziale ed integrale ») si divide in cinque capitoli, dedicati rispettivamente: i) alla nozione di limite (di una funzione, di una successione...) e di continuità; ii) al calcolo differenziale in una variabile, includendovi l'integrazione indefinita come « antiderivazione », e lo studio di semplici equazioni differenziali; iii) alle serie di potenze, con applicazioni al calcolo numerico ed allo studio delle funzioni analitiche; iv) al calcolo differenziale in più variabili; v) all'integrazione delle funzioni di punti, con cenni sull'integrazione secondo Stieltjes (indispensabile in Calcolo delle Probabilità).

Concludendo, pur nel ristretto ambito della matematica « che serve » per gli studenti di Economia e Commercio, l'autore riesce a dare un quadro sommario ma vivace ed acuto della nostra scienza, che certamente stimolerà i migliori di essi ad un fruttuoso approfondimento personale.

FERNANDO BERTOLINI

H. C. VAN DE HULST – *Light Scattering by Small Particles*, pag. 470, 103 figure, 46 tabelle. (J. Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, (1957), $ 12.00.

Lo scopo che si è assunto l'autore di questo libro è stato di raccogliere e di ordinare le molte ricerche fatte, anche nel passato, sulla diffusione delle onde elettromagnetiche, nel campo della radiazione visibile' da parte di particelle di piccole dimensioni e di presentare semplici metodi di calcolo per i vari casi. Ovviamente il problema è classico, ma riallacciandosi al grande problema della difiusione delle onde di varia natura, viene a toccare argomenti che formano oggetto di studio anche nella meccanica quantistica, nella chimica fisica e nella astrofisica. Anzi i metodi che sono stati sviluppati in questi rami della scienza, come ad esempio, l'analisi delle fasi ed i metodi variazionali, trovano utili applicazioni nei problemi di ottica ondulatoria trattati nel libro.

Data l'ampiezza degli argomenti esaminati, l'autore ha voluto restringere molto chiaramente il suo compito: egli tratta la diffusione coerente, singola, da parte di particelle indipendenti. A tale scopo il libro è stato suddiviso in tre parti e 21 capitoli.

Nella prima parte sono ricavati teoremi generali per particelle di dimensioni, forma e composizione arbitrarie e si mostra che la diffusione da una particella finita è pienamente caratterizzata dalla conoscenza di quattro funzioni di ampiezza, dipendenti dalle direzioni della radiazione incidente e diffusa. La conoscenza di queste funzioni permette di ricavare l'intensità e la polarizzazione della radiazione diffusa, le sezioni d'urto delle particelle per diffusione ed assorbimento e la pressione di radiazione esercitata sulla particella dalla radiazione incidente.

La seconda parte è la più estesa del libro: in essa vengono calcolate le funzioni di ampiezza per una grande varietà di particelle. Sono considerate particelle di forma non particolare ma di dimensioni molto grandi o molto piccole rispetto alla lunghezza d'onda, sfere omogenee di grandezza arbitraria, per cui vale la teoria rigorosa di Mie, e particelle di altre forme regolari, come cilindri regolari e corpi con superfici lievemente curve. Naturalmente non manca il classico problema dell'ottica di una goccia d'acqua e dell'arcobaleno. Numerosi grafici e tabelle accompagnano le soluzioni esposte per i vari casi.

La terza parte infine è dedicata alle applicazioni ai vari domini della scienza: alla chimica con lo studio della diffusione nelle soluzioni colloidali, negli aerosoli e nei mezzi anisotropi, alla meteoro-

logia, con lo studio della diffusione ed assorbimento della luce nelle nebbie, nubi e pioggia e dell'attenuazione atmosferica delle microonde, ed infine all'astrofisica con lo studio delle atmosfere planetarie ed interstellari. Un'accurata e ragionata bibliografia segue ogni capitolo.

Il libro, che è mantenuto su di un piano matematico di media levatura e solo negli ultimi capitoli presenta qualche dettaglio sperimentale, è scritto con estrema chiarezza. I vari problemi vengono lucidamente inquadrati e più che affrontare in modo rigoroso gli sviluppi algoritmici, l'autore si preoccupa di sviluppare con dovizia di dati i risultati ottenuti. Per tale motivo il libro può interessare sia il non specialista per una rapida introduzione all'argomento, sia lo specialista per l'aiuto nei calcoli.

G. Poiani

Hugo N. Swenson, J. Edmund Woods – *Physical Science for Liberal Arts Students*, John Wiley e Sons, Inc., Publishers, 1957, pp. VI-333, 138 figure, 39 tavole fotografiche.

Questo libro, destinato agli studenti di scienze umanistiche, si propone di descrivere i principali metodi e i procedimenti più efficaci che sono stati e sono tuttora impiegati nel campo delle scienze naturali e, più specificamente, della fisica, della chimica e della astronomia. Gli autori cercano di interessare allo studio di queste scienze, giovani indirizzati a campi del sapere che richiedono un orientamento culturale assai diverso. Per questo si sforzano di porre in risalto la genesi storica e metodologica con cui sono state acquisite quelle nozioni che possono trovare posto oggi in un trattato scientifico elementare. Gli autori non sono tuttavia riusciti a conciliare in modo soddisfacente questa esigenza con il loro desiderio di dare un quadro completo delle scienze trattate. Il numero troppo grande di argomenti esaminati costringe spesso l'esposizione ad un livello superficiale e non sufficientemente rigoroso, mentre in altri punti l'aspirazione di rendere il trattato di interessante lettura complica inutilmente l'esposizione, introducendo talvolta esempi eccessivamente complessi. Lodevole invece la veste editoriale e le numerose e belle illustrazioni

A. Alberigi

J. Friedel – *Les dislocations* (Gauthier-Villars, Paris (1956), pagine VI-314.

Il libro di Friedel, frutto di un corso di lezioni tenuto dall'autore all'Ecole des Mines di Parigi, rappresenta il più recente tentativo di descrizione unitaria di quanto lo studio teorico delle dislocazioni ha saputo dire riguardo alle loro proprietà e all'influenza che questi difetti esercitano sulle proprietà dei solidi. Il libro è diviso in tre parti. Nella prima vengono descritte le proprietà delle dislocazioni isolate, sia in mezzi elastici continui che in cristalli, e i loro moti conservativi e non conservativi; le proprietà delle dislocazioni parziali nei reticoli compatti sono discusse in notevole dettaglio, ed un capitolo è dedicato alla teoria della crescita dei cristalli dalla fase vapore ed ai modelli proposti per l'origine delle dislocazioni in cristalli cresciuti da soluzione. La seconda parte tratta gli insiemi di dislocazioni e la loro influenza sulle proprietà meccaniche dei solidi: dopo una discussione del reticolo di dislocazioni di Frank e delle proprietà dei bordi dei grani, vengono esposte le idee correnti su indurimento da lavoro, raddolcimento e ricristallizzazione, flusso e microflusso, sfaldatura. Infine, la terza parte tratta l'interazione delle dislocazioni con difetti

puntiformi: particolare attenzione è dedicata all'interazione con impurità e alle conseguenti teorie della durezza come determinata da precipitazione, da soluzione microscopicamente uniforme o da atmosfere di Cottrell. Consistentemente in tutto il libro i risultati della teoria vengono confrontati con i risultati sperimentali disponibili.

La presentazione è in vari punti spiccatamente personale, come l'autore stesso riconosce nella prefazione: alcuni punti rappresentano in realtà contributi originali, sviluppati dall'autore durante la stesura del libro. La trattazione è sempre chiara e fa spesso uso di modelli semplici, cosicchè può essere seguita anche da chi ha solo conoscenze piuttosto rudimentali di fisica dello stato solido. Il libro è perciò adatto soprattutto per chi, anche non avendo interessi diretti di ricerca nel campo, desidera acquistare una conoscenza più precisa della interpretazione corrente delle propriteà meccaniche dei solidi in termini di dislocazioni. Alla completezza del libro come esposizione generale dello stato presente della teoria delle dislocazioni, e come guida utile a chi desideri fare ricerche nel campo, avrebbe giovato un atteggiamento più critico verso la materia ed una trattazione più dettagliata della teoria matematica delle dislocazioni (ad esempio, il modello di Peierls-Nabarro è a mala pena citato.

M. P. Tosi



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